Floris Takens Seminars

Two examples of fast-slow dynamics in mechanical engineering

Nonlinear passive vibration control

and

Transient phenomena in reed musical instruments

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Plan

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Nonlinear Energy Sink (NES)

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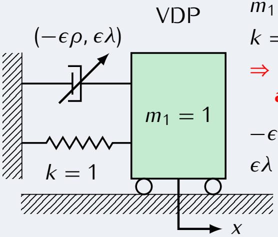
Targeted Energy Transfer (TET)

[Vakakis *et al.* (2006), Springer]

- Used for **passive** and **broadband** vibration mitigation in mechanical and acoustic systems:
  - Free vibrations
  - Forced vibrations
  - Self-sustained vibrations

Self-sustained oscillations: Van der Pol (VDP) oscillator

Van der Pol (VDP) oscillator



$m_1 = 1$: **mass**

$k = 1$: **stiffness**

$\Rightarrow \omega_0 = \sqrt{k/m_1} = 1$: **angular frequency**

$-\epsilon\rho$: **linear** damping

$\epsilon\lambda$: **nonlinear** damping

$$\boxed{\ddot{x} - \epsilon\rho\dot{x} + \epsilon\lambda\dot{x}x^2 + x}$$

ρ : **bifurcation parameter**

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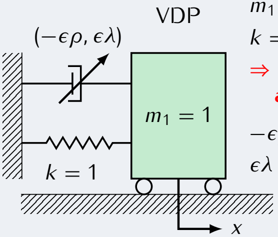

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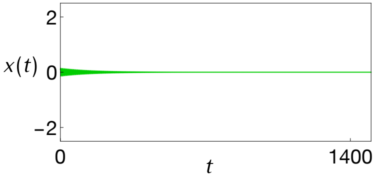
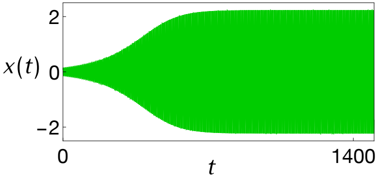
$\boxed{\rho = 0}$: **Hopf bifurcation point** of equilibrium $x^e = 0$

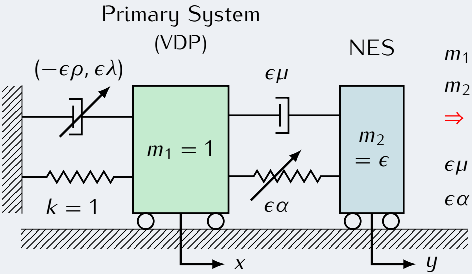
- $\boxed{\rho < 0}$: **Stable** equilibrium



- $\boxed{\rho > 0}$: **Unstable** equilibrium + **periodic solution**



Van der Pol oscillator coupled to an NES



$m_1 = 1$: **mass** of the VdP

$m_2 = \epsilon$: **mass** of the NES

$\Rightarrow \epsilon = m_2/m_1$: **mass ratio between NES and VDP**

$\epsilon\mu$: linear damping of the NES

$\epsilon\alpha$: cubic stiffness of theNES

x: displacement of the VDP y: displacement of the NES

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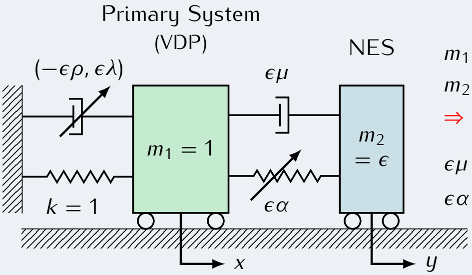

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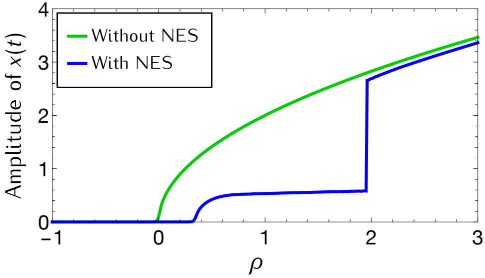
Assumption

Small-mass NES $\Rightarrow \boxed{0 < \epsilon \ll 1}$

Mitigation limit of the NES

Bifurcation diagram

Steady-state amplitude as a function of the **bifurcation parameter ρ**



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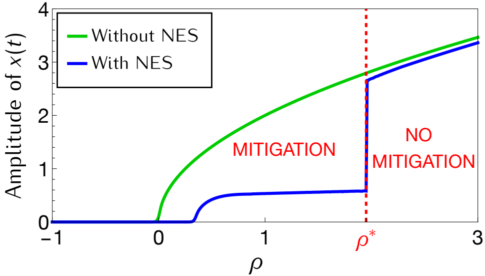

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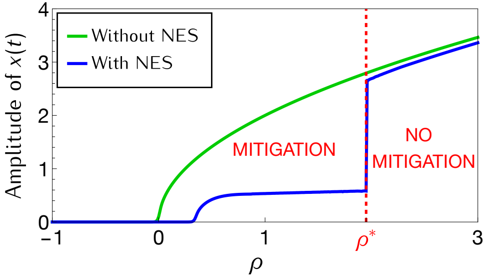

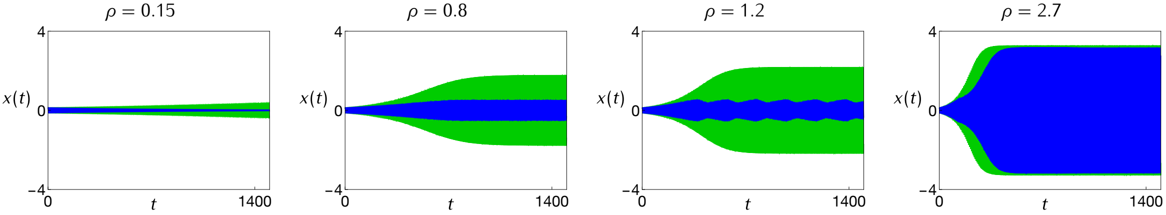

Stabilization (**linar** effect)

Periodic regime (**nonlinear** effect)

Quasi-periodic regime (SMR) (**nonlinear** effect)

Periodic regime (with high amplitude)

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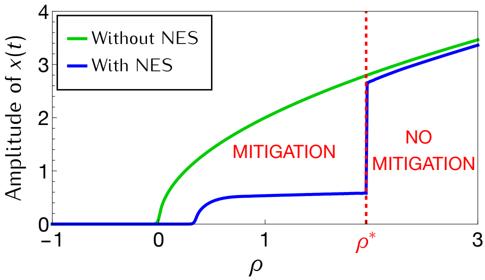

Zeroth-order global stability analysis [Gendelman & Bar (2012), Physica D]

Theoretical prediction of the mitigation limit when $\boxed{\epsilon = 0}$

Equations of the Amplitude-Phase Modulation Dynamics (APMD)

- Change of variable: x (VDP) and y (NES) $\Rightarrow$ $\boxed{u = x + \epsilon y}$ and $\boxed{v = x - y}$

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$\equiv$ u et v are **amplitude-** and **phase-modulated** $\Rightarrow$ $\boxed{u(t) = r(t)\sin(t + \theta_1(t))}$ et $\boxed{v(t) = s(t)\sin(t + \theta_2(t))}$

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$\hookrightarrow$ Computing the APMD using a **perturbation technique**

$$\boxed{\begin{aligned} \dot{r} &= \epsilon f(r, s, \Delta) \\ \dot{s} &= g_1(r, s, \Delta, \epsilon) \\ \dot{\Delta} &= g_2(r, s, \Delta, \epsilon) \end{aligned}}$$

r et s: amplitudes of u and v

$\Delta = \theta_1 - \theta_2$: phase difference between u and v

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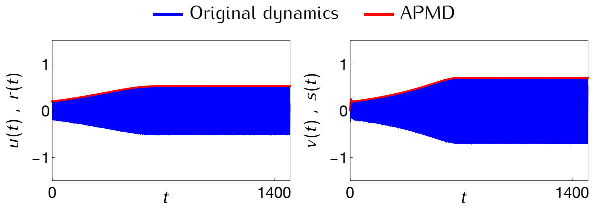
Original dynamics: **Periodic regime** $\equiv$ **APMD:** **Non-zero equilibrium**



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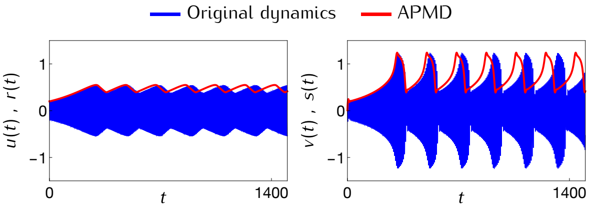

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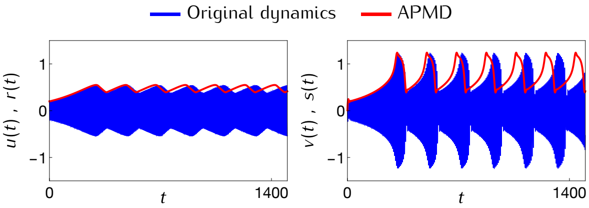

Original dynamics: SMR $\equiv$ **APMD:** **Periodic regime**

APMD $\equiv$ fast-slow dynamical system : **2 fast variables** s **and** Δ et **1 slow variable** r

$\Rightarrow$ Time evolution of the system = succession **fast epochs** and **slow epochs**

Zeroth-order fast-slow analysis of the APMD

APMD ≡ fast-slow dynamical system

- Time evolution of the system = succession **fast epochs** and **slow epochs**
- Theoretical analysis:
  - [Gandelman & Bar (2012), Physica D]: **multiple scales method**
  - [Bergeot *et al.* (2016), Int J Non Linear Mech]: **Geometric Singular Perturbation Theory (GSPT)**

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APMD
at the **fast time scale** t

$$\dot{r} = \epsilon f(r, s, \Delta)$$
$$\dot{s} = g_1(r, s, \Delta, \epsilon)$$
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APMD
at the **slow time scale** $\tau = \epsilon t$

$$r' = f\,(r, s, \Delta)$$
$$\epsilon s' = g_1\,(r, s, \Delta, \epsilon)$$
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$$\boxed{\begin{aligned} \dot{r} &= 0 \\ \dot{s} &= g_1(r,s,\Delta,0) \\ \dot{\Delta} &= g_2(r,s,\Delta,0) \end{aligned}}$$

$\hookrightarrow$ **fast subsystem**
describes the fast epochs

We sate $\boxed{\epsilon = 0}$

Singularly perturbed system

APMD
at the **slow time scale** $\tau = \epsilon t$

$$\begin{aligned} r' &= f(r,s,\Delta) \\ \epsilon s' &= g_1(r,s,\Delta,\epsilon) \\ \epsilon \Delta' &= g_2(r,s,\Delta,\epsilon) \end{aligned}$$

$$\boxed{\begin{aligned} r' &= f(r,s,\Delta) \\ 0 &= g_1(r,s,\Delta,0) \\ 0 &= g_2(r,s,\Delta,0) \end{aligned}}$$

$\hookrightarrow$ **slow subsystem**
describes the slow epoch

Zeroth-order fast-slow analysis of the APMD

Critical manifold (CM)

$$\boxed{\mathcal{M}_0 = \left\{ (r, s, \Delta) \,\middle|\, g_1(r, s, \Delta, 0) = 0 \,,\ g_2(r, s, \Delta, 0) = 0 \right\}}$$

$\boxed{r = H(s)}$ and $\boxed{\Delta = G(s)}$

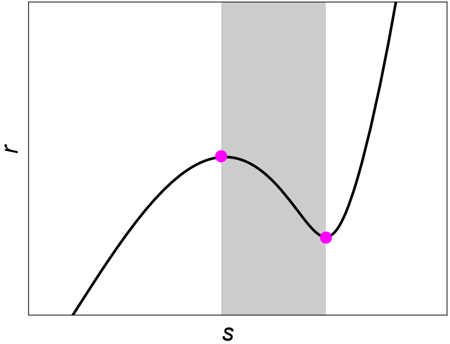

Figure. — $\boxed{r = H(s)}$

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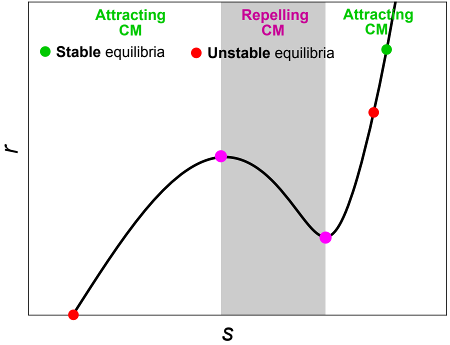


Figure. — $\boxed{r = H(s)}$

⇒From the fast subsystem: Stability $\mathcal{M}_0 \Rightarrow$ 2 attracting branches et 1 repelling branch

⇒From the slow subsystem: Equilibria (on $\mathcal{M}_0$) ⇒ • **Stable** equilibria • **Unstable** equilibria

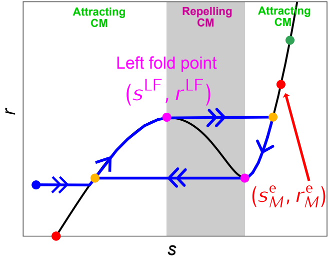
Zeroth-order fast-slow analysis of the APMD

Global stability analysis: theoretical prediction of the mitigation limit

• Initial condition • **Stable** equilibria • **Unstable** equilibria • Fold points • Zeroth-order arrival point

Original dynamics (OD): SMR

APMD: Relaxation oscilllations



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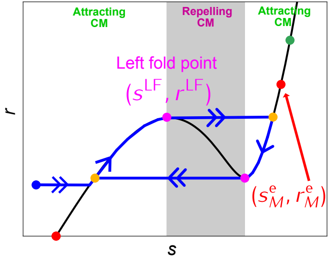

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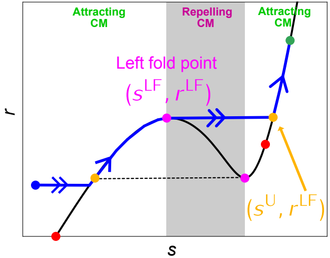

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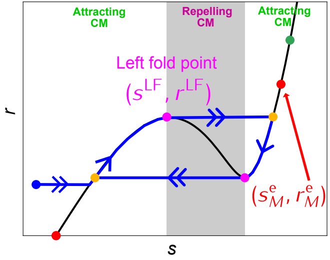

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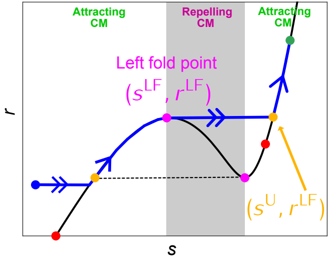

Zeroth-order arrival point

$$\boxed{(s^{\mathrm{a}}, r^{\mathrm{a}}) = (s^{\mathrm{U}}, r^{\mathrm{LF}})}$$

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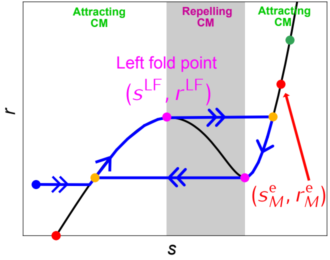

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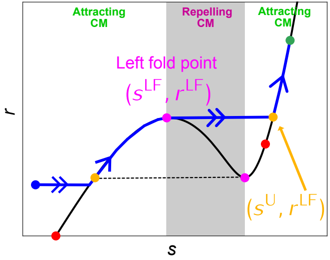

Zeroth-order arrival point

$$(s^{a}, r^{a}) = (s^{U}, r^{LF})$$

Zeroth-order theoretical prediction of the mitigation limit

Value of the bifurcation parameter ρ (denoted as $\boxed{\rho_0^*}$) solution of:

$$\boxed{r^{e}_{M} = r^{a} = r^{LF}} \quad \Rightarrow \text{Analytical expression of } \rho_0^*$$

Zeroth-order fast-slow analysis of the APMD

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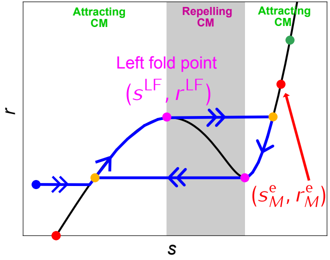

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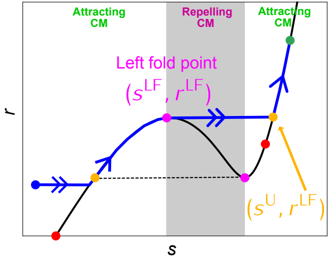


Zeroth-order arrival point

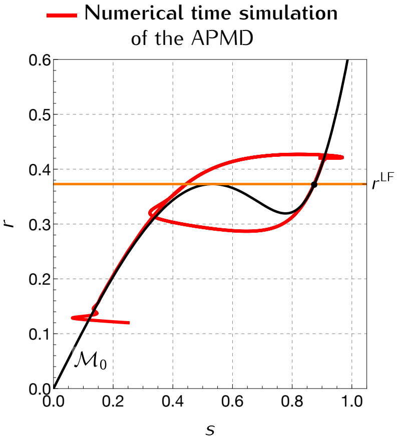
$$\boxed{(s^a, r^a) = (s^U, r^{LF})}$$

Today: presentation of 2 orignal results

- **Result 1:** **scaling law** and new theoretical estimation of the mitigation limit [Bergeot (2021), J Sound Vib]
- **Result 2:** Dynamics of a **VDP coupled to a bistable NES** [Bergeot (2024), Physica D]

Plan

The limitations of zeroth-order analysis – theoretical vs numerical results for $\boxed{\epsilon = 0.015}$



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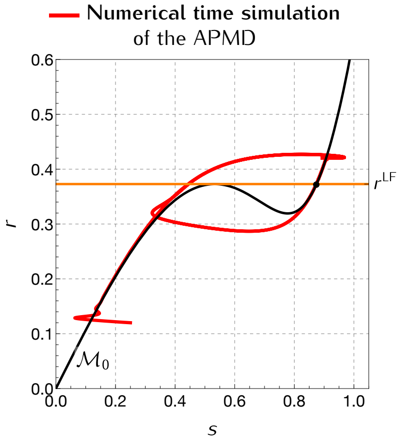

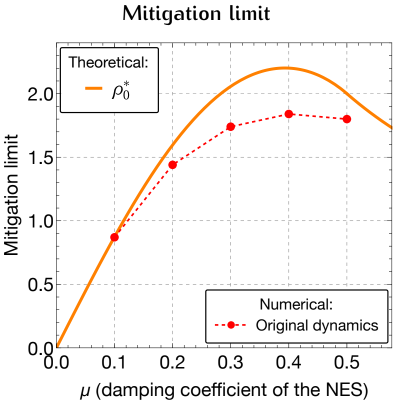

- For "large" values of ϵ: Underestimation of the arrival point $\Rightarrow$ Overestimation of the mitigation limit

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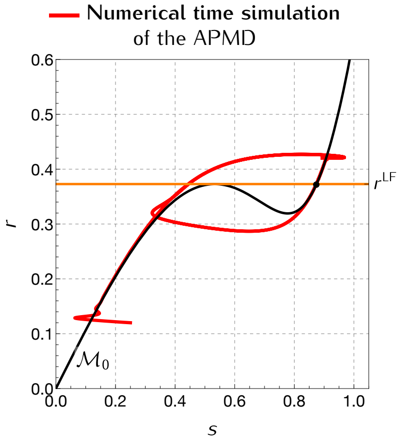
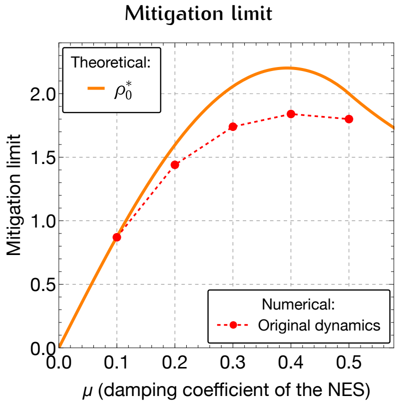
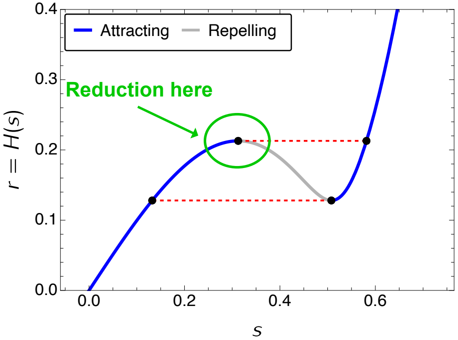




- For "large" values of ϵ: Underestimation of the arrival point $\Rightarrow$ Overestimation of the mitigation limit
- No description of the evolution of the mitigation limit as a function of ϵ.

Center manifold reduction of the APMD at the left fold point and scaling law (1/2)



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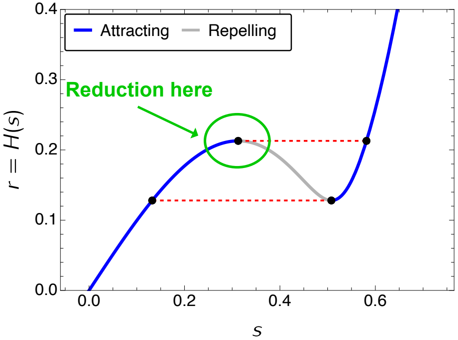

At the **left fold point** $(r^{\mathrm{LF}}, s^{\mathrm{LF}}, \Delta^{\mathrm{LF}})$ the APMD ...

$$\boxed{\begin{aligned} r' &= f(r, s, \Delta) \\ \epsilon s' &= g_1(r, s, \Delta, \epsilon) \\ \epsilon \Delta' &= g_2(r, s, \Delta, \epsilon) \end{aligned}}$$

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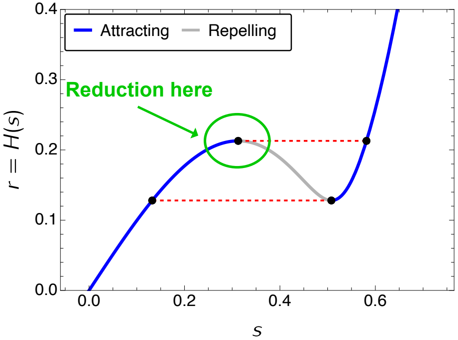

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... is reduced to the normal form of the **dynamic saddle-node bifurcation**:

$$\begin{aligned} \hat{\epsilon} x' &= x^2 + y \\ y' &= 1 \end{aligned}$$

y: new slow variable linked to r
x: new fast variable linked to s et Δ
$\hat{\epsilon}$: new small parameter linked to ϵ

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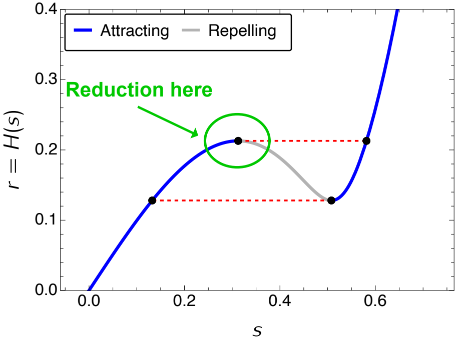

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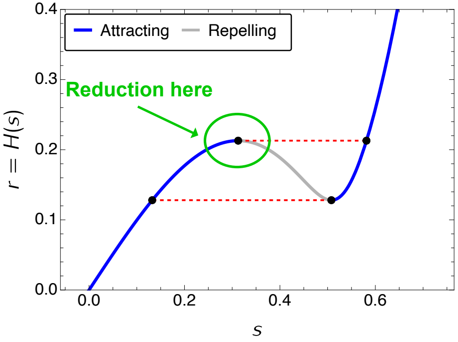

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$\Rightarrow$ **Has a analytical solution**:

Scaling law (normal form)

Analytical expression of **x as a function y and $\hat{\epsilon}$**:

$$\boxed{x^\star(y, \hat{\epsilon}) = \hat{\epsilon}^{1/3} \frac{\mathrm{Ai}'\left(-\hat{\epsilon}^{-2/3} y\right)}{\mathrm{Ai}\left(-\hat{\epsilon}^{-2/3} y\right)}}$$

Ai: Airy function

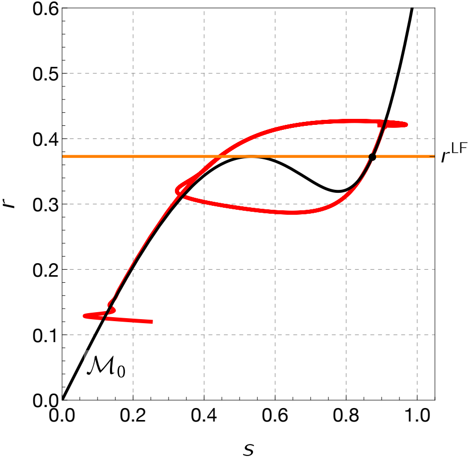
Center manifold reduction of the APMD at the left fold point and scaling law (2/2)

Scaling law (APMD)

Analytical expression of s **as a function of** r **and** ϵ:

$$\boxed{s^\star(r,\epsilon) = s^{\mathrm{LF}} + \epsilon^{1/3} K_1 \frac{\mathrm{Ai}'\left(-\epsilon^{-2/3} K_2 \left(r - r^{\mathrm{LF}}\right)\right)}{\mathrm{Ai}\left(-\epsilon^{-2/3} K_2 \left(r - r^{\mathrm{LF}}\right)\right)}}$$

- K_1 and K_2: constants depending on model parameters
- Ai and Ai$'$: Airy function and its derivative



Center manifold reduction of the APMD at the left fold point and scaling law (2/2)

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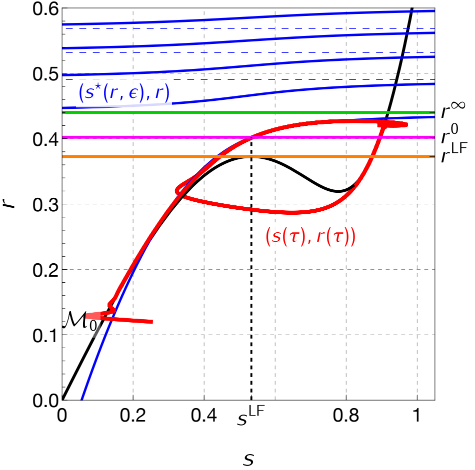

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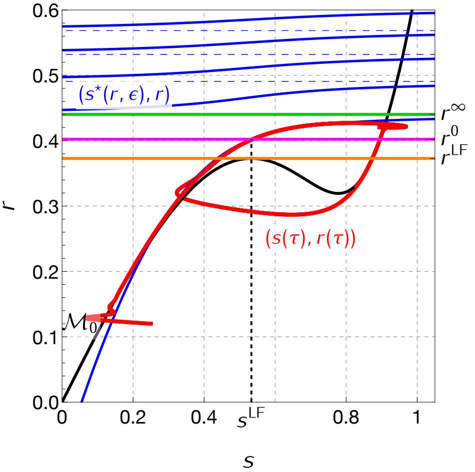
- K_1 and K_2: constants depending on model parameters
- Ai and Ai′: Airy function and its derivative

New estimation of the arrival point (s^A, r^A)

$$\boxed{r^0 < r^a < r^\infty}$$

r^0: defined as $s^\star(r) = s^{\mathrm{LF}}$ $\Rightarrow$ first zero of Ai′

r^∞: defined as $s^\star(r) \to \infty$ $\Rightarrow$ first zero of Ai



New theoretical estimation of the mitigation limit

From the zeroth-order analysis

Value of ρ (denoted as $\boxed{\rho_0^*}$) solution of:

$$\boxed{r_M^{\mathrm{e}} = r^{\mathrm{a}} = r^{\mathrm{LF}}}$$

From the scaling law

Lower bound: $\boxed{\rho_{\epsilon,\mathrm{inf}}^*}$ solution of:

$$\boxed{r_M^{\mathrm{e}} = r^{\mathrm{a}} = r^{\infty}}$$

Upper bound: $\boxed{\rho_{\epsilon,\mathrm{sup}}^*}$ solution of:

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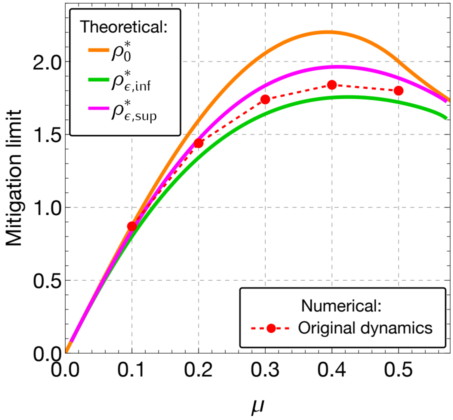
As a function of μ for $\boxed{\epsilon = 0.015}$:



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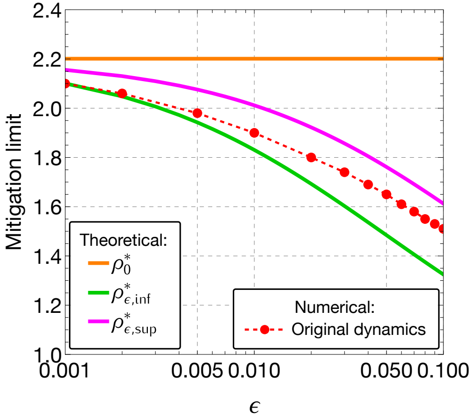
Upper bound: $\boxed{\rho_{\epsilon,\sup}^*}$ solution of:

$$\boxed{r_M^{\mathrm{e}} = r^{\mathrm{a}} = r^{0}}$$

As a function of ϵ for $\boxed{\mu = 0.4}$:



Plan

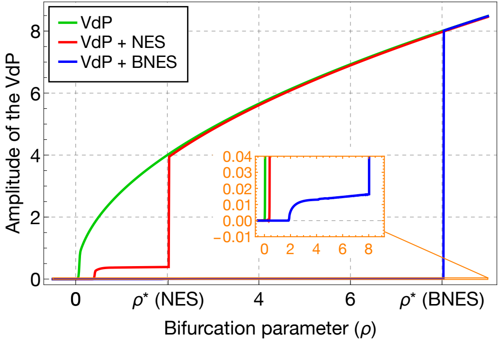
Bistable Nonlinear Energy Sink (BNES)

BNES = **cubic NES** with in addition a **negative linear stiffness** element:

$$\boxed{\ddot{y} + \mu\dot{y} - \beta y + \alpha y^3 = 0}$$

- Zero equilibrium $\boxed{y_0^e = 0}$ **unstable**
- **2 stable** non-zero equilibria:
  - Right equilibrium: $\boxed{y_1^e = \sqrt{\frac{\beta}{\alpha}}}$
  - Left equilibrium: $\boxed{y_2^e = -\sqrt{\frac{\beta}{\alpha}}}$

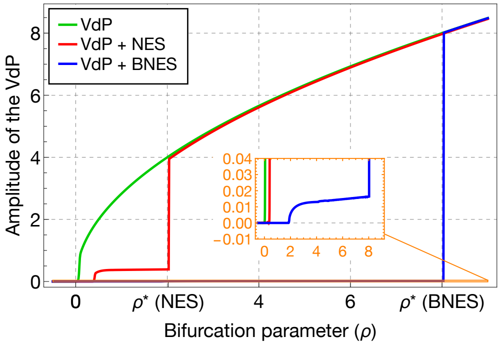
NES vs BNES



Bifurcation diagram

- $\boxed{\rho^*(\text{NES}) \ll \rho^*(\text{BNES})}$
- Very low amplitude attenuation regimes with BNES

NES vs BNES



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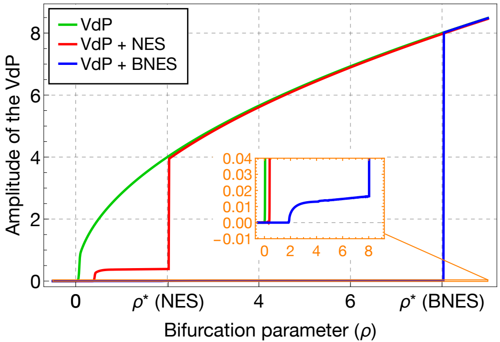
⚠ **Robustness**

NES vs BNES



Bifurcation diagram

- $\boxed{\rho^*(\text{NES}) \ll \rho^*(\text{BNES})}$
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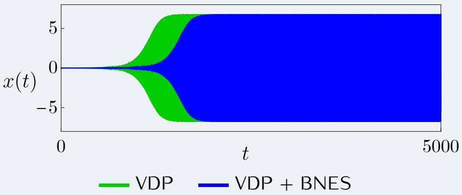
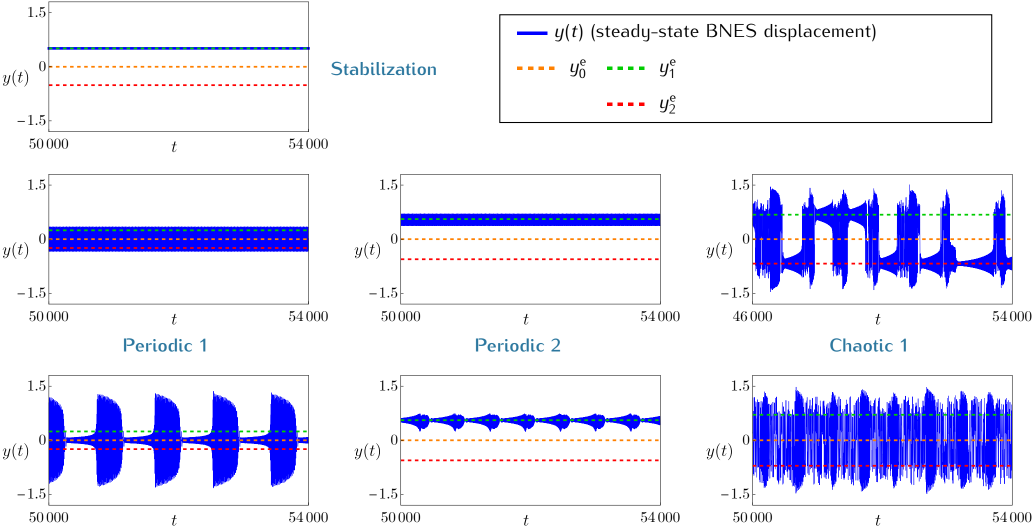
⚠ Robustness

Identification of the regimes

$\boxed{\rho > \rho^*}$: No mitigation (periodic)



$\boxed{\rho < \rho^*}$: 7 ATTENUATION REGIMES



Stabilization

Periodic 1 — Periodic 2 — Chaotic 1

SMR 1 — SMR 2 — Chaotic 2

Zeroth-order fast-slow analysis of the APMD

$\Rightarrow$ 1 : 1 **resonance capture assumption**

- $\boxed{u(t) = r(t)\sin(t + \theta_1(t))}$
- $\boxed{v(t) = b(t) + s(t)\sin(t + \theta_2(t))}$

$\hookrightarrow$ **Perturbation technique** $\rightarrow$ APMD:

$$\boxed{\begin{aligned} \dot{r} &= \epsilon f(a, c, \delta) \\ \dot{b} &= g_1(b, c, \epsilon) \\ \dot{s} &= g_2(a, b, c, \delta) \\ \dot{\Delta} &= g_3(a, b, c, \delta, \epsilon) \end{aligned}}$$

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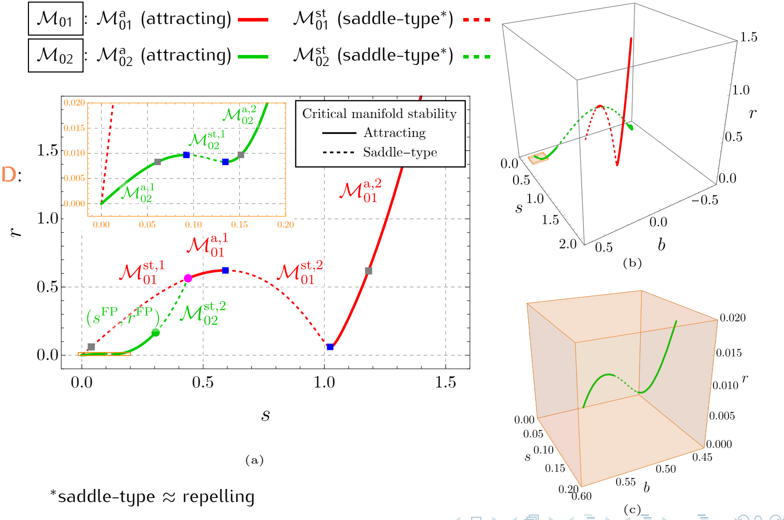
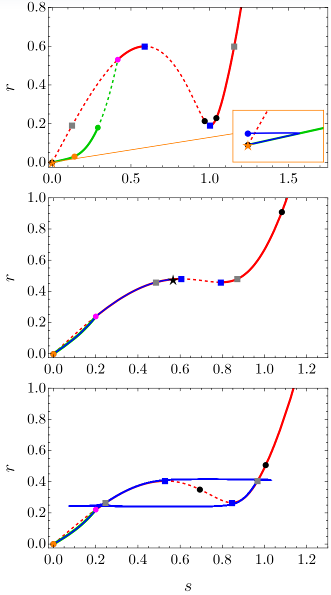
The critical manifold $\mathcal{M}_0$ has two main branches:

$\boxed{\mathcal{M}_{01}}$: $\mathcal{M}_{01}^{\mathrm{a}}$ (attracting) $\mathcal{M}_{01}^{\mathrm{st}}$ (saddle-type*)

$\boxed{\mathcal{M}_{02}}$: $\mathcal{M}_{02}^{\mathrm{a}}$ (attracting) $\mathcal{M}_{02}^{\mathrm{st}}$ (saddle-type*)



*saddle-type ≈ repelling

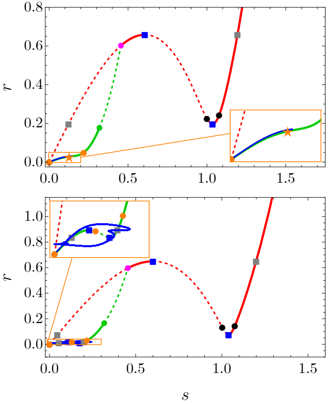


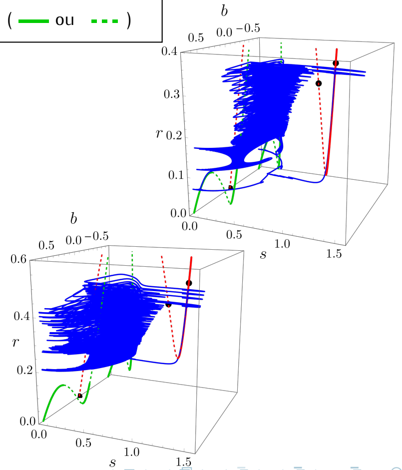
In the (s, r)-plane:

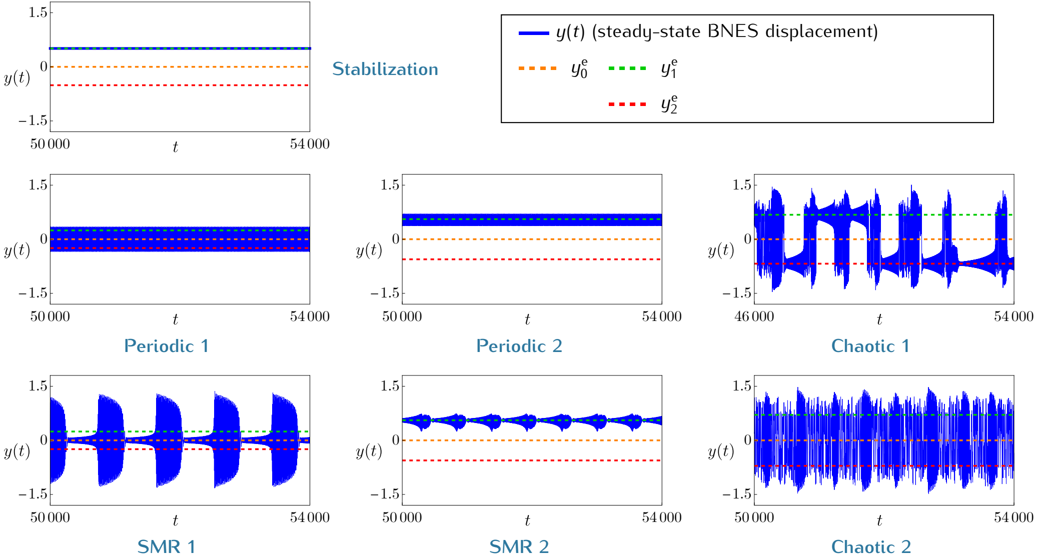
Trajectory of the APMF (numerical simulation)

$\star$ stable et $\bullet$ unstable equilibria on $\mathcal{M}_{01}$ (ou)

$\star$ stable et $\bullet$ unstable equilibria on $\mathcal{M}_{02}$ (ou)






$y(t)$ (steady-state BNES displacement)
y_0^e
y_1^e
y_2^e
1.5
0
−1.5
$y(t)$
t
50 000
54 000
46 000
Stabilization
Periodic 1
Periodic 2
Chaotic 1
SMR 1
SMR 2
Chaotic 2

Plan

Effect of noise on the mitigation limit of a cubic NES

- **Numerical study (Monte Carlo):** [Bergeot (2023), Int. J. Non-Linear Mech.]
  $\Rightarrow$ **Noise tends to promote the non mitigation regimes for high noise levels**
- **Analytical study**: PhD of Israa Zogheib (Nov. 2023- ; Dir. Nils Berglund and Baptiste Bergeot)
  $\Rightarrow$ **Study of a reduced problem:** normal form of a **dynamic saddle-node bifurcation** with **noise acting on the slow variable**

Self-sustained oscillator connected to a BNES

- Finding and studying other solutions of the fast subsystem (such as periodic, quasiperiodic or even chaotic motions)
- Global stability analysis: computing the basins of attraction of all the solutions of the fast subsystem

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Plan

Plan

Single-reed musical instruments:

Saxophones

Clarinets

- Modeled by nonlinear dynamical systems linking **control parameters** (**mouth pressure γ**, **lip force F**) to **output variables** (**acoustic pressure p inside the mouthpiece**)
- Previous theoretical studies on sound production performed with **control parameters constant in time** show that:
  - Appearance of sound = **Hopf bifurcation of the trivial equilibrium** (**silence, i.e., $p = 0$**) to a **stable periodic solution** (**musical note**)
  - Several stable solutions coexist in general = **Multistability**

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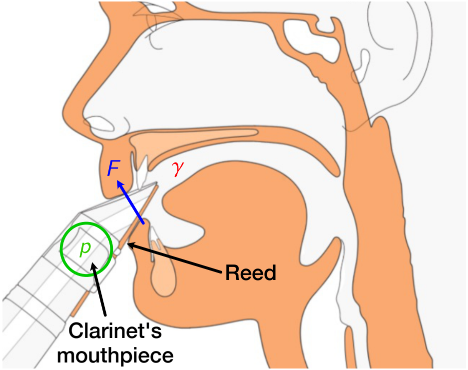
Saxophones

Clarinets



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F: force applied by the lip on the reed

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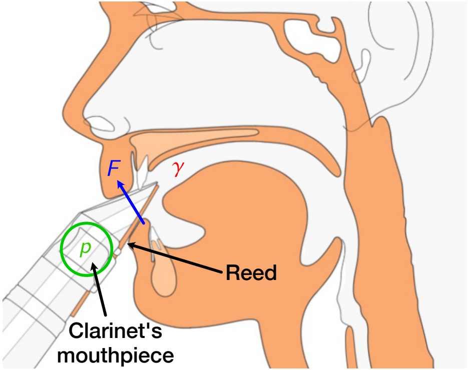

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Observation

During transient phases the musician varies the control parameters in time

Questions

- In the context of musical acoustics: during an attack transient, how can the dynamic characteristics of the control parameters be related to:
  1. the appearance of sound?
  2. the nature of the sound in case of multistability? ⇒ silence? note? another note?
- Open problems in nonlinear dynamics: nonlinear dynamical systems with time-varying parameters when
  1. a bifurcation point is crossed ⇒ bifurcation delay [Benoi et al. (1991), Lect. Notes Math.]
  2. a multistability domain is crossed ⇒ rate-induced tipping [Ashwin et al. (2012), Philos. Trans. R. Soc. Lond. A]

Presented work

Predicting appearance of sound and the nature of the sound produced (i.e., tipping or not) in simple models in the case of a slow linear variation of the control parameter mouth pressure γ

$$\dot{\gamma} = \epsilon \quad \text{with} \quad \boxed{0 < \epsilon \ll 1}$$

ϵ: rate of change

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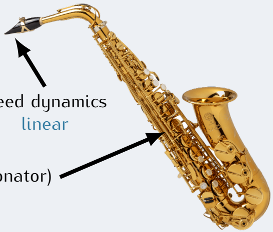
ϵ: rate of change

Refined physical model

Air flow vs p
nonlinear (Bernoulli)



Reed dynamics
linear

Body (linear acoustic resonator)

Refined physical model

Air flow vs p
nonlinear (Bernoulli)

Reed dynamics
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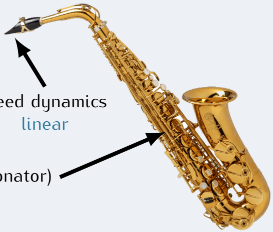
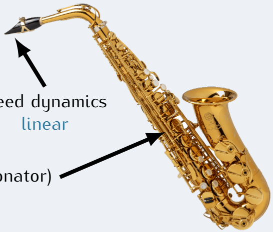


$\Rightarrow$ System of coupled nonlinear ODEs

Refined physical model

Air flow vs p
nonlinear (Bernoulli)



Reed dynamics
linear

Body (linear acoustic resonator)

$\Rightarrow$ System of coupled nonlinear ODEs

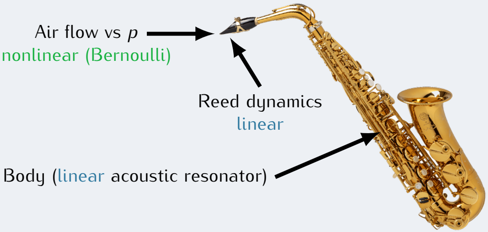
Simplest model having bistability

$\Rightarrow$ One-dimensional ODE:

$$\boxed{\dot{x} = f(x, \gamma)}$$

x: amplitude of the mouthpiece pressure p
γ: control (or bifurcation) parameter

Refined physical model



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Simplest model having bistability

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- Silence: $\boxed{x = 0}$
- Musical note: $\boxed{x = \text{constant}}$

Model with a slowly time-varying γ = fast-slow system

$$\boxed{\begin{aligned} \dot{x} &= f(x, \gamma) \\ \dot{\gamma} &= \epsilon \end{aligned}}$$

x: **fast variable**

γ: **slow variable**

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Simple model
at the
fast time scale t

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Simple model
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slow time scale $\tau = \epsilon t$

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Simple model at the fast time scale t

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$\hookrightarrow$ fast subsystem

We state

$$\boxed{\epsilon = 0}$$

Simple model at the slow time scale $\tau = \epsilon t$

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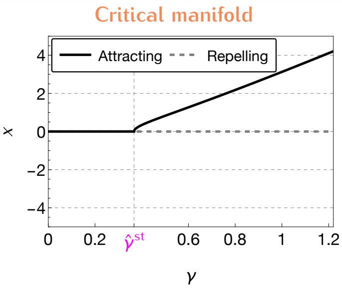
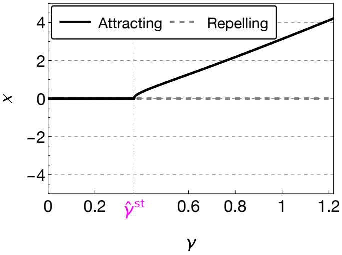
Plan

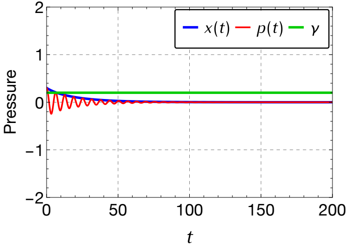


$\hat{\gamma}^{st}$: **Static** bifurcation point

Critical manifold



γ (constant) $< \hat{\gamma}^{st}$: Silence



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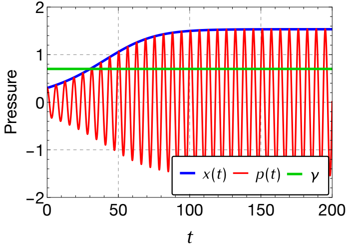

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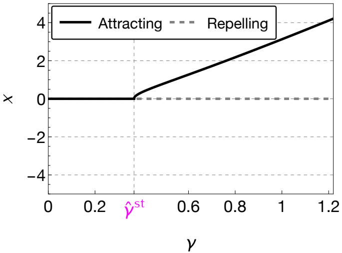

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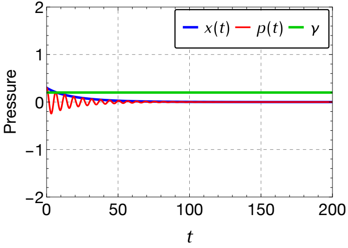

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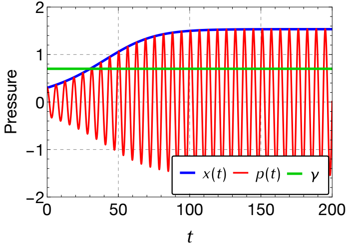

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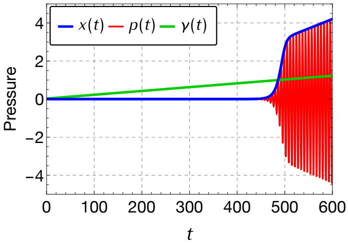
γ slowly varies in time



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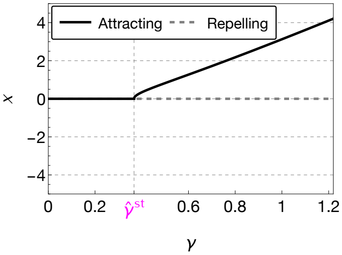

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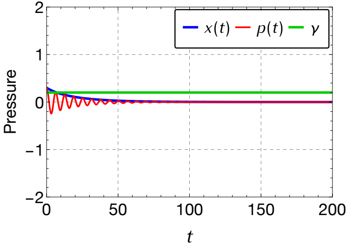

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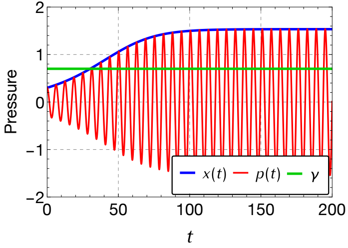

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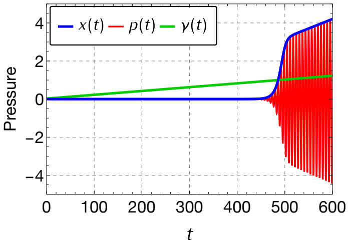
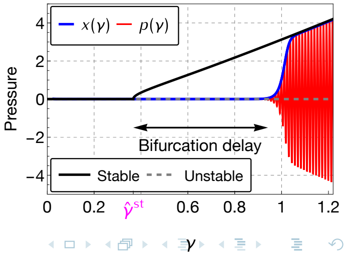


$x(t)$ and $p(t)$ as a function of $\gamma(t)$



The need for stochastic modelling

Noise (physical or numerical) **reduces bifurcation delay** and must be **taken into account** in the models

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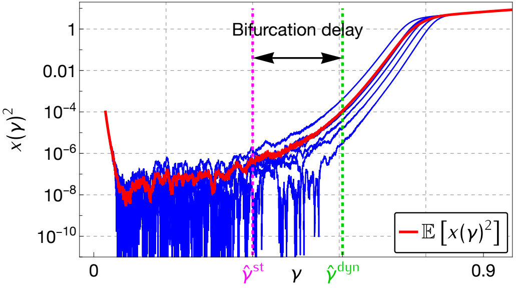

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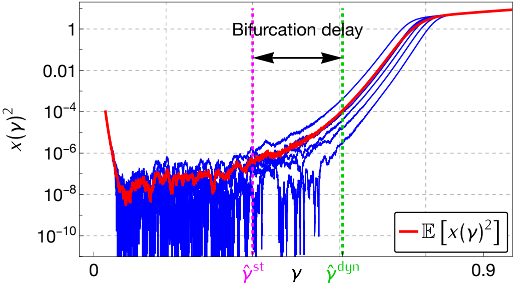

Definition: dynamic bifurcation point $\hat{\gamma}^{dyn}$

Value of γ such as $\boxed{\mathbb{E}\left[x(\gamma)^2\right] = x(\gamma_0)^2}$

Analytical prediction of bifurcation delay

Analytical solution of:

$$\dot{x} = f(x, \gamma) + \sigma\xi(t) \approx a(\gamma)x + \sigma\xi(t)$$
$$\dot{\gamma} = \epsilon$$

[Bergeot & Vergez (2022), Nonlinear Dyn]

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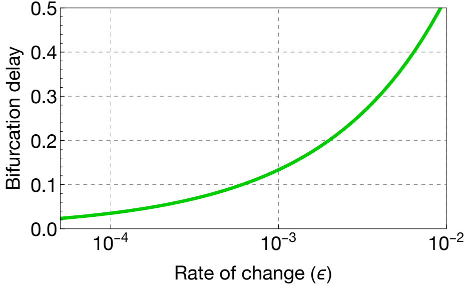

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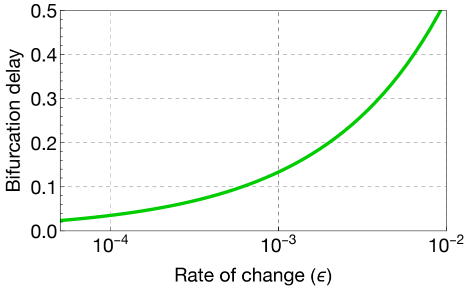
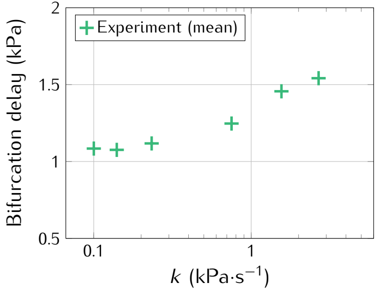


Experimental: as a function $k \propto \epsilon$:



[Bergeot *et al.* (2014), J Acoust Soc Am]

Plan

Deterministic model first

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x: **fast variable**

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Remark. $f(x,\gamma)$ now takes into account that **reed motion is limited by the instrument mouthpiece**

Critical manifold

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$$\boxed{\mathcal{M}_0 = \left\{(x,\gamma) \in \mathbb{R}^2 \mid f(x,\gamma) = 0\right\}}$$

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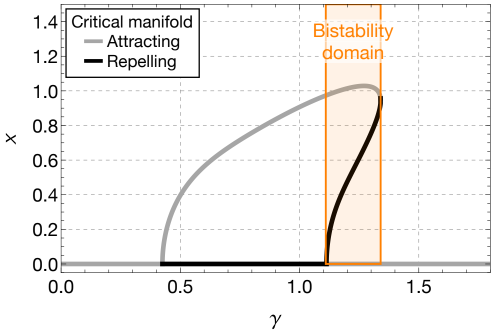

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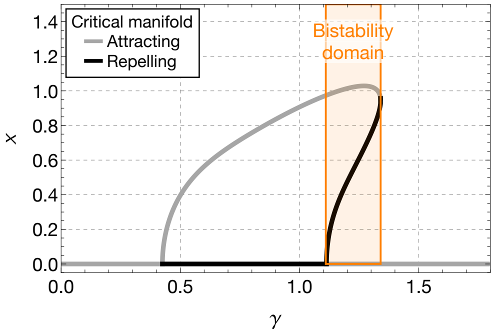

In the bistability domain the critical manifold has:

- 2 attracting branches
- **1 repelling branch**

Problem statement

$$\boxed{\begin{aligned}\dot{x} &= f(x, \gamma)\\ \dot{\gamma} &= \epsilon\end{aligned}} \qquad (1)$$

- For a given initial condition, which attracting branch of the critical manifold will the trajectory of (1) follow when it crosses the bistability domain?

$\Rightarrow$ More concisely: **tipping of not tipping?**

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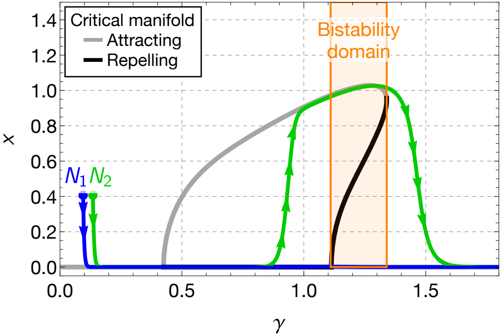

Figure. Numerical simulations of (1) with two close initial conditions N_1 and N_2

Observation

Although N_1 and N_2 are very close in the phase space, they lead to **qualitatively different behaviors during transient**:

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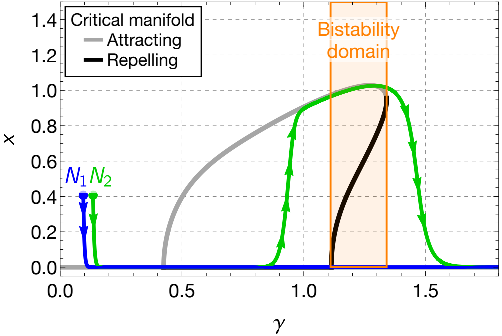

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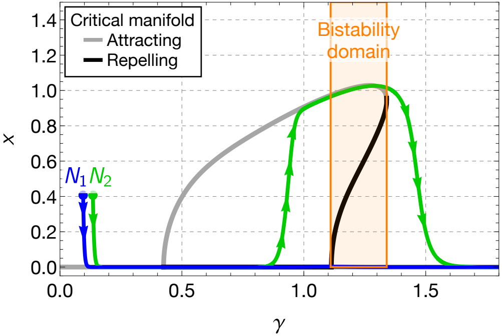

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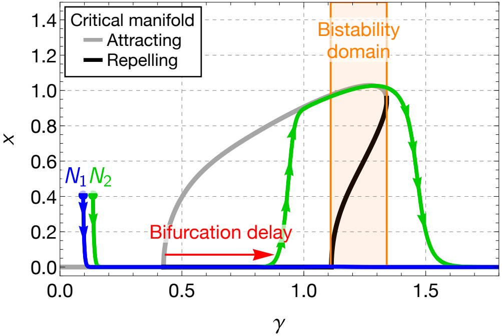

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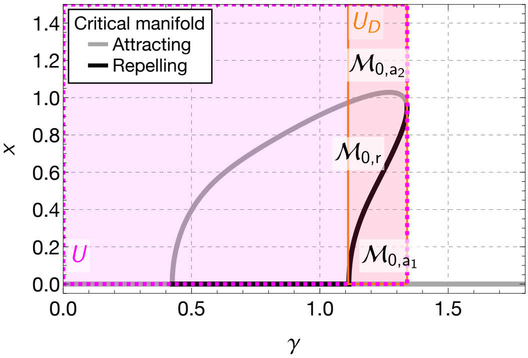
Remark

Bifurcation delay

$$\begin{cases} \dot{x} = f(x, \gamma) \\ \dot{\gamma} = \epsilon \end{cases} \tag{1}$$

$\boxed{U_D = (\gamma_l, \gamma_u) \times \mathbb{R}^+}$; $\boxed{U = (0, \gamma_u) \times \mathbb{R}^+}$



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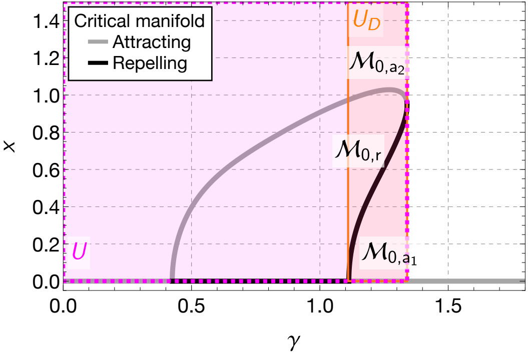

In U_D, $\mathcal{M}_0$ has **3 branches**:

$$\boxed{\mathcal{M}_{0,a_i} = \{(x,\gamma) \in U_D \mid x = x_i^\star(\gamma)\}\,, \quad i = 1,2}$$

$$\boxed{\mathcal{M}_{0,r} = \{(x,\gamma) \in U_D \mid x = x_3^\star(\gamma)\}}$$

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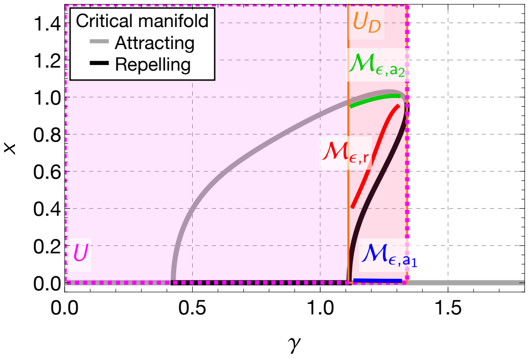

In U_D, $\mathcal{M}_0$ has 3 **branches**:

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Fenichel's theorem

$\Downarrow$

In U_D, one has **3 invariant manifolds**:

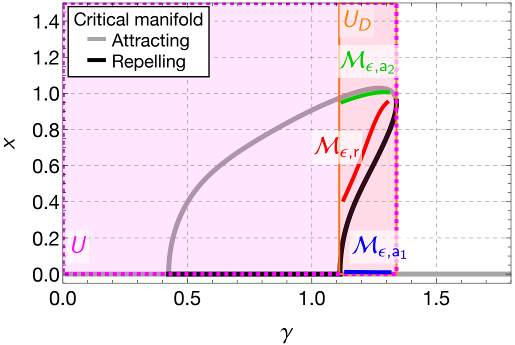
$$\boxed{\mathcal{M}_{\epsilon,a_1} = \{(x,\gamma) \in U_D \mid x = \bar{x}_1(\gamma,\epsilon)\}}$$

$$\boxed{\mathcal{M}_{\epsilon,a_2} = \{(x,\gamma) \in U_D \mid x = \bar{x}_2(\gamma,\epsilon)\}}$$

$$\boxed{\mathcal{M}_{\epsilon,r} = \{(x,\gamma) \in U_D \mid x = \bar{x}_3(\gamma,\epsilon)\}}$$

Tipping separatrix

$$\mathcal{M}_{\epsilon,\mathrm{r}} = \{(x,\gamma) \in U_D \mid x = \bar{x}_3(\gamma,\epsilon)\}$$



Tipping separatrix

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We define the **special solution** S, called **tipping separatrix***, in U as

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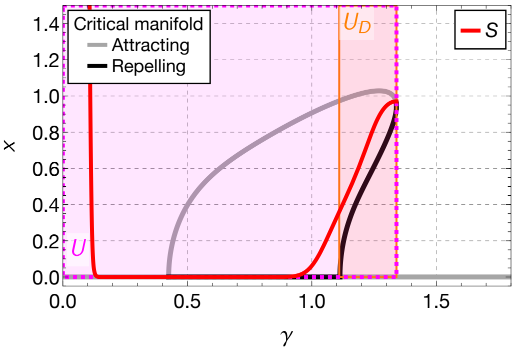

*[Bergeot *et al.* (2024), Chaos]

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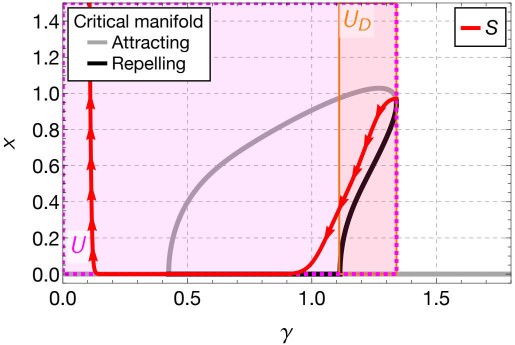
In practice

S is numerically approximated using a time reversal procedure since here $\mathcal{M}_{\epsilon,\mathrm{r}}$ is attracting in reverse time



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Result [Bergeot *et al.* (2024), Chaos]

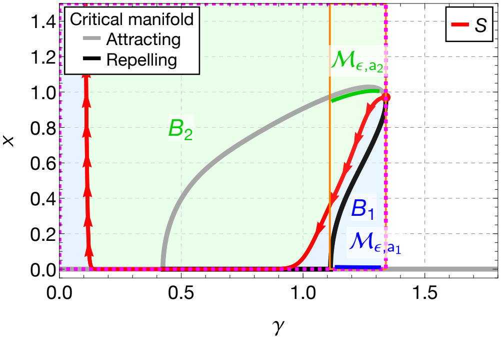
Tipping or not tipping?

The **tipping separatrix** S splits U into two subsets B_1 and B_2:

$$\boxed{B_1 = \{(x, \gamma) \in U \mid x < \bar{x}_3(\gamma, \epsilon)\}}$$ NO TIPPING

$$\boxed{B_2 = \{(x, \gamma) \in U \mid x > \bar{x}_3(\gamma, \epsilon)\}}$$ TIPPING

Orbits originating from initial conditions in B_1 (resp. B_2) follow $\mathcal{M}_{\epsilon,a_1}$ (resp. $\mathcal{M}_{\epsilon,a_2}$) when the slow variable γ crosses the bistability domain U_D



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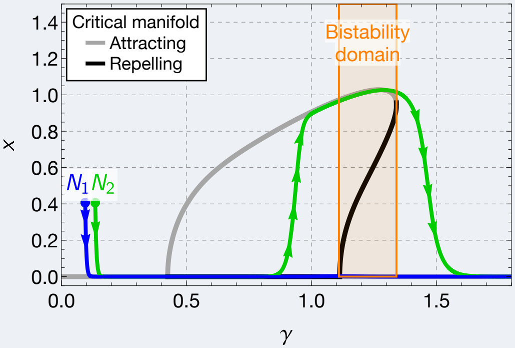
$$\boxed{B_2 = \{(x, \gamma) \in U \mid x > \bar{x}_3(\gamma, \epsilon)\}} \quad \text{TIPPING}$$

Orbits originating from initial conditions in B_1 (resp. B_2) follow $\mathcal{M}_{\epsilon,a_1}$ (resp. $\mathcal{M}_{\epsilon,a_2}$) when the slow variable γ crosses the bistability domain U_D

Back to the problem statement



Result [Bergeot *et al.* (2024), Chaos]

Tipping or not tipping?

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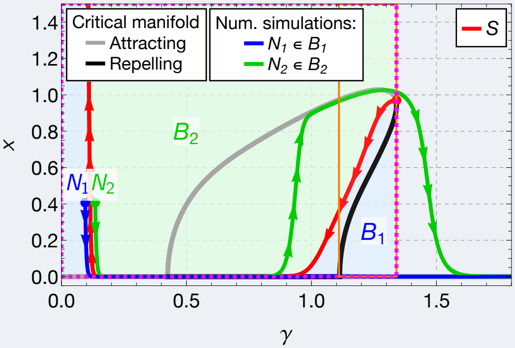

Explanation. Although N_1 and N_2 are very close in the phase space, **they are not in the same B subset**, that leads to **qualitatively different behavior during transient**

Plan

Multistability in more refined models of reed instruments

- Equivalent of the tipping separatrix in the case of a bistability between musical notes
- Compute separatrices using advanced numerical methods: continuation, machine learning

Effect of noise

- The tipping separatrix implies bifurcation delay:
  - The effect of noise must be taken into account

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Thank you for your attention

Questions?

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