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Passive mitigation of self-sustained oscillations by means of a bistable nonlinear energy sink

Numerical investigation and fast-slow analysis

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LaMé Laboratoire de Mécanique Gabriel Lamé

Outline

Nonlinear Energy Sink (NES)

- Oscillators with strongly nonlinear stiffness (here purely cubic) with linear damping:

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BNES = **cubic NES** with a negative linear stiffness element:

$$\boxed{\ddot{y} + \mu\dot{y} - \beta y + \alpha y^3 = 0}$$

- Zero equilibrium $\boxed{y_0^{e} = 0}$ **unstable**
- **2 stable** non-zero equilibria: $\boxed{y_1^{e} = \sqrt{\frac{\beta}{\alpha}}}$ $\boxed{y_2^{e} = -\sqrt{\frac{\beta}{\alpha}}}$

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Connected to a Primary Structure (PS), the NES (or BNES):

- can adjust its frequency to that of the PS (relation amplitude/frequency)
- irreversibly absorbs the energy of the PS (under certain conditions)

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Targeted Energy Transfer - TET

Used for passive and broadband vibration mitigation in mechanical and acoustic systems:

- Free vibrations
- Forced vibrations
- Self-sustained vibrations

Self-sustained oscillations: Van der Pol (VdP) oscillator

Self-sustained oscillations

Creation and sustaining of an oscillating motion (periodic, quasi-periodic, chaotic, ...) by a non-oscillating source of energy

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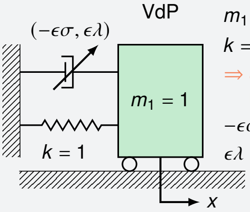

- $m_1 = 1$: **mass**
- $k = 1$: **stiffness**
- $\Rightarrow \omega_0 = \sqrt{k/m_1} = 1$: **natural angular frequency**
- $-\epsilon\sigma$: **linear** damping
- $\epsilon\lambda$: **nonlinear** damping

$$\boxed{\ddot{x} - \epsilon\sigma\dot{x} + \epsilon\lambda\dot{x}x^2 + x}$$

σ : bifurcation parameter

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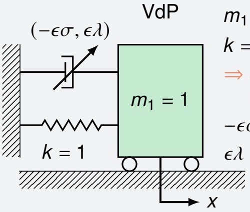

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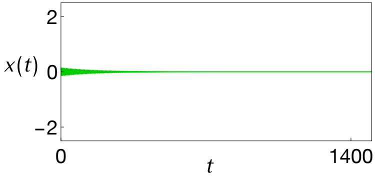
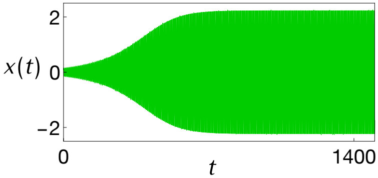
$\boxed{\sigma = 0}$: Hopf bifurcation point of equilibrium $x^e = 0$

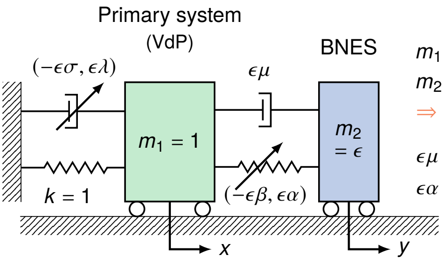
- $\boxed{\sigma < 0}$: **Stable** equilibrium



- $\boxed{\sigma > 0}$: **Unstable** equilibrium + periodic solution



Van der Pol oscillator connected to a BNES



$m_1 = 1$: **mass** of the VdP

$m_2 = \epsilon$: **mass** of the BNES

$\Rightarrow \epsilon = m_2/m_1$: **mass ratio between the BNES and the VdP**

$\epsilon\mu$: linear damping of the BNES

$\epsilon\alpha$: cubic stiffness of the BNES $\quad -\epsilon\beta$: negative stiffness of the BNES

x: displacement of the VdP $\quad y$: displacement of the BNES

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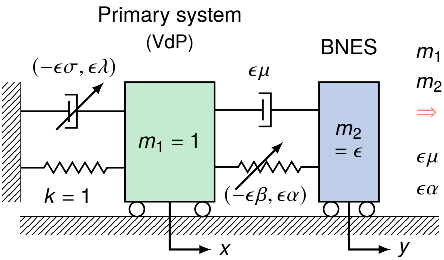

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x: displacement of the VdP y: displacement of the BNES

Small-mass BNES $\Rightarrow \boxed{0 < \epsilon \ll 1}$

Outline

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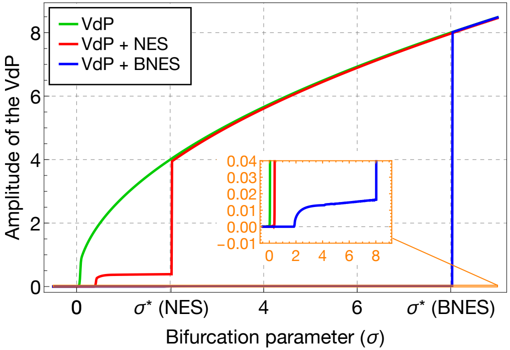
NES vs BNES

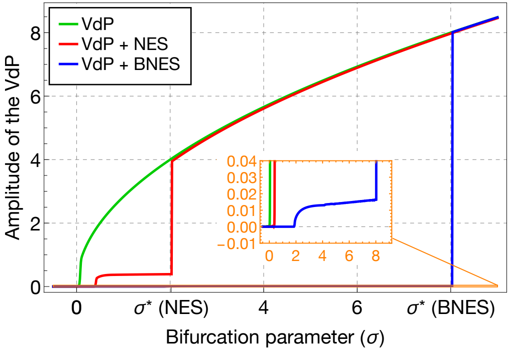


Bifurcation diagram

- $\boxed{\sigma^*(\text{NES}) \ll \sigma^*(\text{BNES})}$
- Very low amplitude attenuation regimes with BNES

σ^*: mitigation limit

NES vs BNES



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⚠ Robustness

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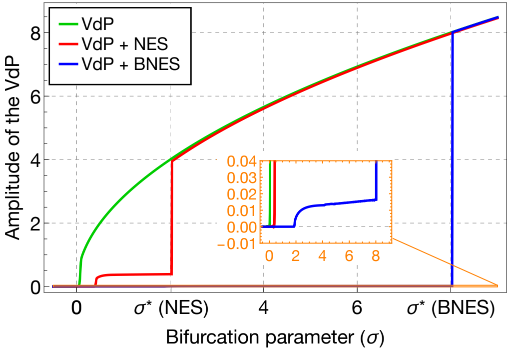

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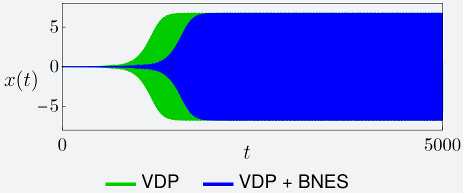
⚠ Robustness

Identification of the regimes

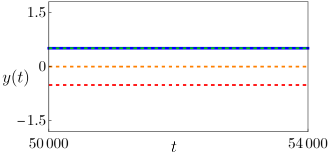
$\boxed{\sigma > \sigma^*}$: No mitigation (periodic)



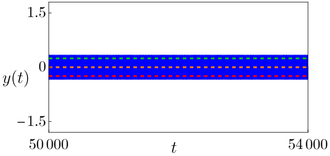
$\sigma < \sigma^*$: 7 MITIGATION REGIMES

$y(t)$ (steady-state BNES displacement)

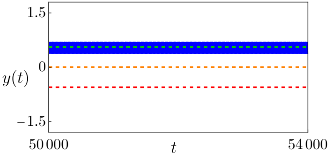
y_0^e y_1^e y_2^e



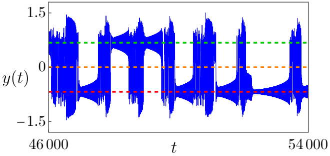
Stabilization



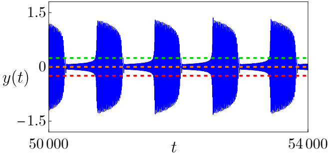
Periodic 1



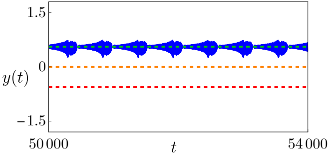
Periodic 2



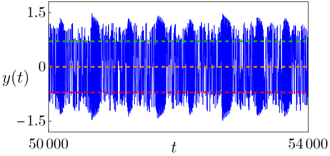
Chaotic 1



SMR1



SMR2



Chaotic 2

Outline

Equations of the Amplitude-Phase Modulation Dynamics (APMD)

- Change of variables : x (VDP) and y (BNES) $\Rightarrow$ $\boxed{u = x + \epsilon y}$ and $\boxed{v = x - y}$

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  $\hookrightarrow$ Use of the Multiple Scale/Harmonic Balance Method (MSHBM)*, modified to consider that:
  - u (VdP motion): zero-mean periodic-like motion $\Rightarrow$ $\boxed{u(t) = a(t)\sin(t + \theta_1(t))}$
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* [Luongo & Zulli (2012), Nonlinear Dynamics]

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$$\boxed{\begin{aligned}\dot{a} &= \epsilon f(a, c, \delta)\\ \dot{b} &= g_1(b, c, \epsilon)\\ \dot{c} &= g_2(a, b, c, \delta)\\ \dot{\delta} &= g_3(a, b, c, \delta, \epsilon)\end{aligned}}$$

$\delta = \theta_1 - \theta_2$: phase difference between u and v

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Original dynamics:		**APMD:**
Periodic solution	$\rightarrow$	Equilibrium
Quasiperiodic (SMR)	$\rightarrow$	Periodic solutions

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$\Rightarrow$ APMD $\equiv$ (3,1)-fast-slow : 1 slow variable a and 3 fast variables b, c and δ

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APMD
at the fast time scale t

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APMD
at the slow time scale $\tau = \epsilon t$

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$\hookrightarrow$ fast subsystem

We state $\boxed{\epsilon = 0}$

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$\hookrightarrow$ slow subsystem

Critical manifold - *Also called* ***Slow Invariant Manifold*** *(SIM)*

Obtained by **solving the algebraic part** of the slow subsystem:

$$\boxed{\mathcal{M}_0 = \left\{ (a, b, c, \delta) \in \mathbb{R}^{+^3} \times [-\pi, \pi] \,\middle|\, g_1(b, c, 0) = 0\ ,\ g_2(a, b, c, \delta) = 0\ ,\ g_3(a, b, c, \delta, 0) = 0 \right\}}$$

Fast-slow analysis

Fast-slow analysis

From the fast subsystem: **computing the stability of $\mathcal{M}_0$**: does it attract or repel the fast dynamics?

$\Rightarrow$ Attracting branches: $\mathcal{M}_0^a$ (they attract) and Saddle-type branches: $\mathcal{M}_0^{st}$ (they repel)

From the slow subsystem: **computing its equilibrium solutions** (on $\mathcal{M}_0$)

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Possible asymptotic behavior (when $\epsilon \longrightarrow 0$) of APMD

When points of $\mathcal{M}_0$ are **the only solutions of the fast subsystem**:

- During fast epochs: trajectories outside $\mathcal{M}_0$ towards an attracting branch
- During slow epochs: on a attracting branch of $\mathcal{M}_0$ to a stable equilibrium or moving away from an unstable equilibrium of the slow subsystem

Critical manifold

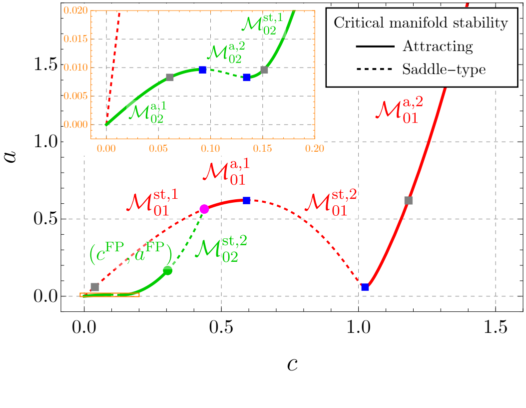
The critical manifold $\mathcal{M}_0$ has two main branches:

$\mathcal{M}_{01}$:

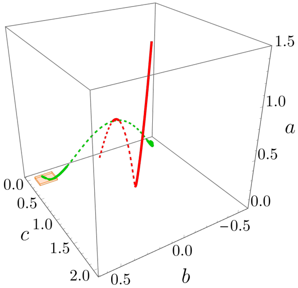
- $\mathcal{M}_{01}^{a}$ (attracting)
- $\mathcal{M}_{01}^{st}$ (saddle-type)

$\mathcal{M}_{02}$:

- $\mathcal{M}_{02}^{a}$ (attracting)
- $\mathcal{M}_{02}^{st}$ (saddle-type)



(a)



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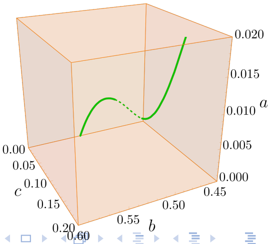

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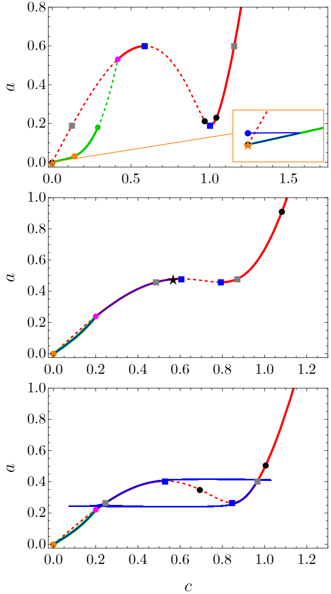
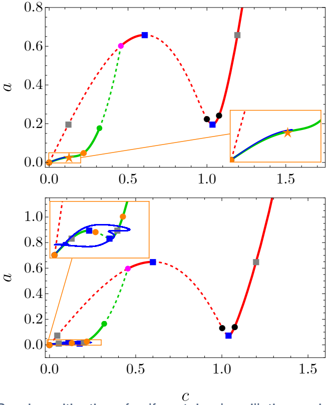
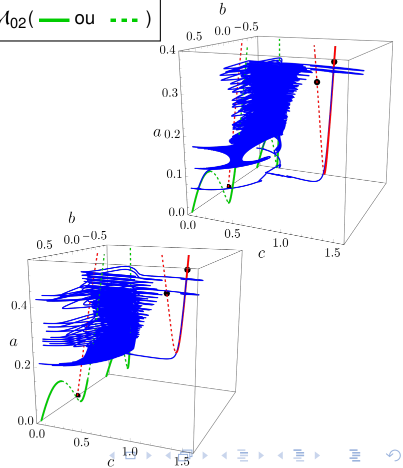
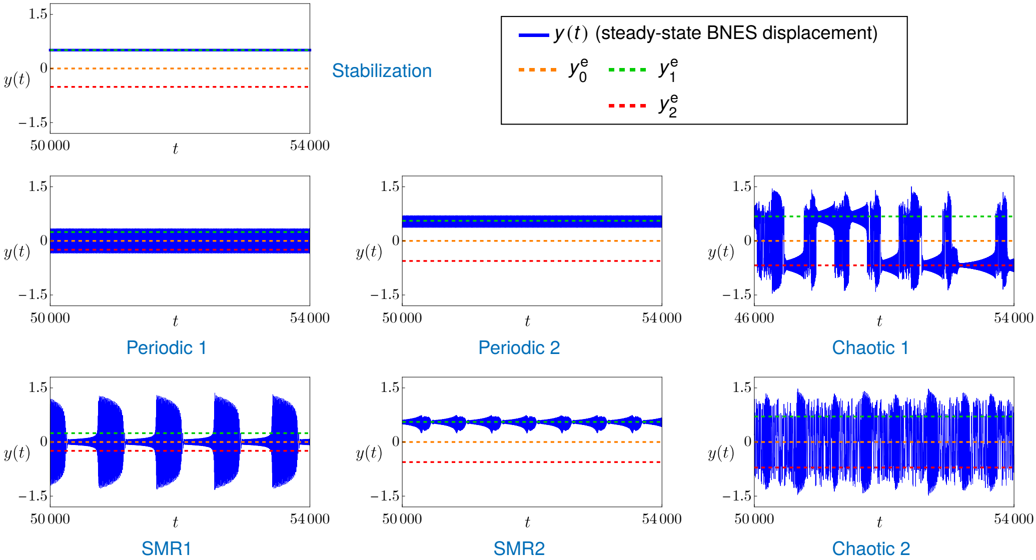


In the (c, a)-plane and the (b, c, a)-space:

- Trajectory of the APMD (numerical simulation)
- ★ stable and ● unstable fixed points on $\mathcal{M}_{01}$ (ou)
- ★ stable and ● unstable fixed points on $\mathcal{M}_{02}$(ou)






$y(t)$ (steady-state BNES displacement)
y_0^e
y_1^e
y_2^e
Stabilization
Periodic 1
Periodic 2
Chaotic 1
SMR1
SMR2
Chaotic 2
$y(t)$
t
1.5
0
−1.5
50 000
54 000
46 000

Outline

Conclusion

$\Rightarrow$ Results from:

[Bergeot and Berger (2024), *Physica D: Nonlinear Phenomena*]

- Computation of an APMD taking into account the specific nature of the BNES (nonzero-mean motion)
- Fast-slow analysis of the APMD enables us to predict or at least interpret the regimes observed on numerical simulations of the initial system

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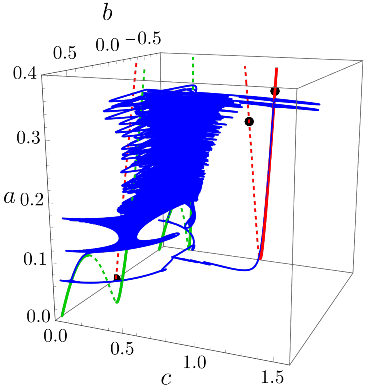
Perspectives

- Finding and studying other solutions of the fast subsystem (such as periodic, quasiperiodic or even chaotic motions)
- Computing the slow invariant manifolds tracking these solutions



Chaotic 1

Thank you for your attention

Questions?

Appendices

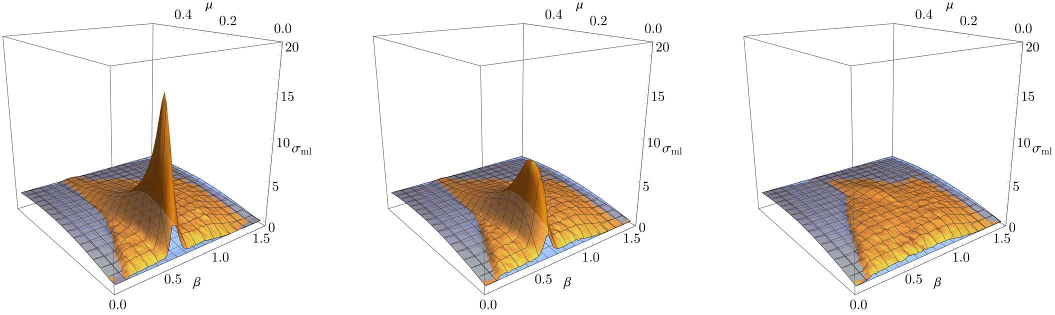
Comparison: VdP + NES vs VdP + BNES

Influence of the initial conditions

σ_{ml} vs (β, μ)

NES: $(x_0, y_0) = (0, 0) + (per, per)$

BNES: $(x_0, y_0) = p_1^e + (per, per)$



$per = 0.001$

$per = 0.01$

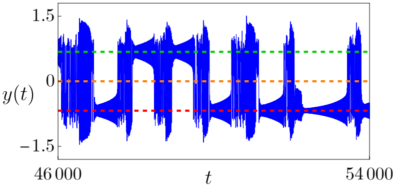
$per = 0.1$

- The BNES can be very efficient but this is not robust
- A little perturbation can switch the system from harmless to harmful situations
- It is always possible to find a set of parameters for which the BNES performs better than the NES

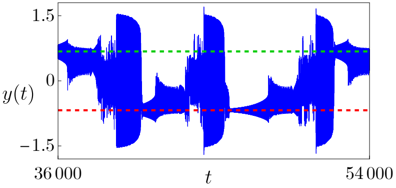
Chaotic 1

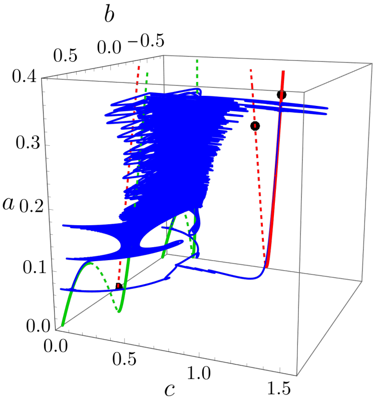
In the (b, c, a)-space

Original dynamics



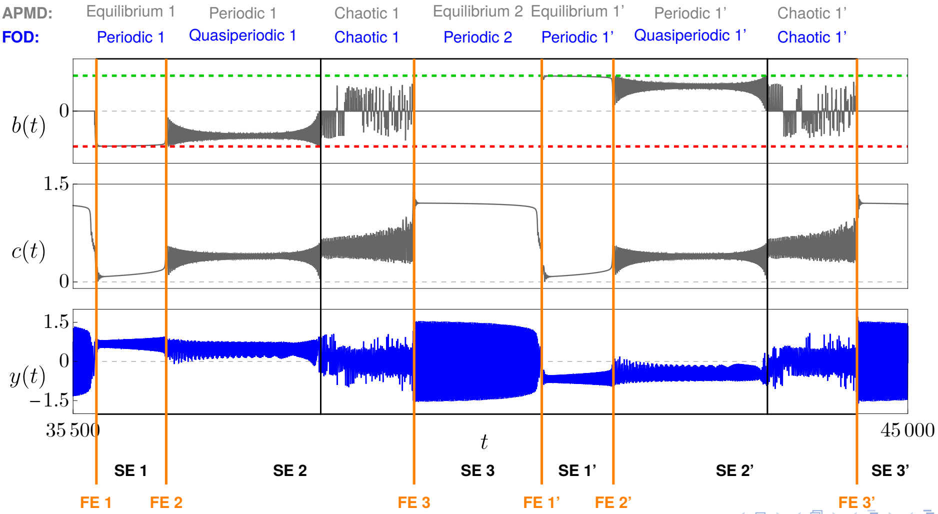
APMD





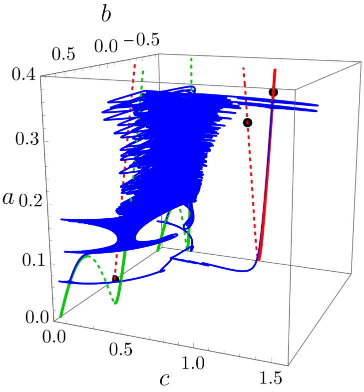
Chaotic 1

The phases of a complete cycle

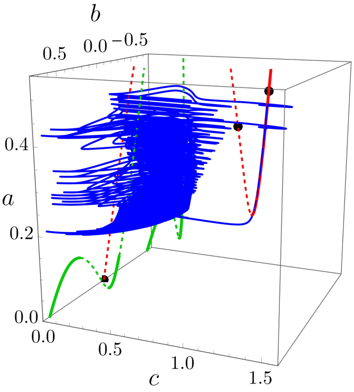


Chaotic 1 and Chaotic 2

In the (b, c, a)-space



Chaotic 1



Chaotic 2