Fluid Structure Interaction

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M2RI: Electronics and Mechanical Engineering

Course name: Acoustics and Vibrations

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Chapter 1

Introduction

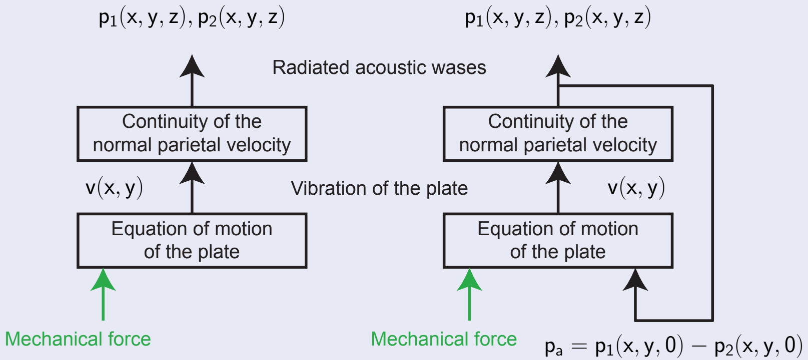
Some references

1. Miguel C. Junger and David Feit, ***Sound, Structures, and Their Interaction, Second Edition***, The MIT Press, 1986.
2. Anders Nilsson and Bilong Liu, ***Vibro-Acoustics (Volume 1 and Volume 2)***, Springer, 2014.
3. Clive L. Dym and Irving H. Shames, ***Solid Mechanics (A variational approach, Augmented Edition)***, Springer, 2013.
4. Catherine Potel, Michel Bruneau, ***Acoustique Générale - équations différentielles et intégrales, solutions en milieux fluide et solide, applications***, Ed. Ellipse, 2006
5. Jean-Claude Pascal, ***Handout of fluid structure interactions lecture***, Université du Maine (Le Mans, France), http://perso.univ-lemans.fr/~jcpascal/Cours/ENSIM3A_Master2_Vibrocoustique.pdf

6. Catherine Potel, Michel Bruneau, ***Acoustique Générale - équations différentielles et intégrales, solutions en milieux fluide et solide, applications***, Ed. Ellipse, 2006

Vibroacoustic coupling: general view

Coupled and **uncoupled** problem

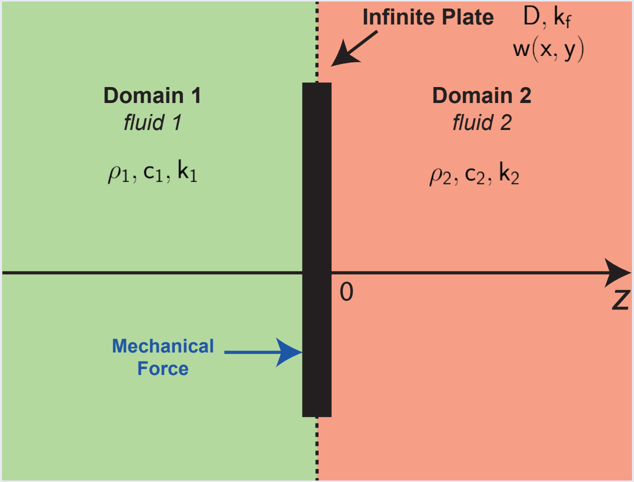


Uncoupled problem
(***light fluid*** : air, ...)

Coupled problem
(***heavy fluid*** : water, ...)

Vibroacoustic coupling: Case of an infinite plate

Diagram of the general problem



Outline

1. **Introduction**

2. **Wave equation in plate and fluid**
   - Bending waves in thin plates
   - Acoustical waves in semi-infinite space

3. **Vibroacoustic coupling: case of an infinite plate**
   - Vibroacoustic coupling: general view
   - Acoustic radiation of the infinite plate
   - Radiation from infinite point-excited plates

Chapter 2

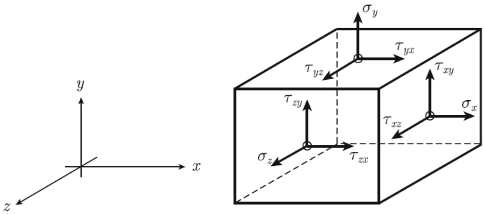
Wave equation in plate and fluid

Chapter 2

Wave equation in plate and fluid

1. Bending waves in thin plates

Hooke's law for isotropic materials



Stress tensor

- **Normal stresses**

  σ_x, σ_y, σ_z

- **Shear stresses**

  Second Newton's law $\Rightarrow \tau_{ij} = \tau_{ji}$

  $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$

Hooke's law for isotropic materials

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\mathbf{Q}\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$\Rightarrow$ Mechanical parameters:

E (**Young modulus**)
ν (**Poisson's ratio**)

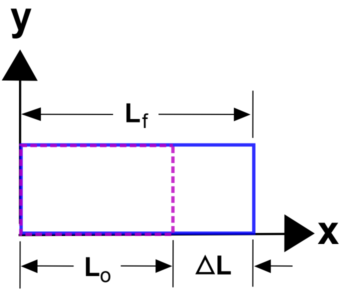
$\Rightarrow$ Strains with respect to the displacements u_x, u_y and u_z:

$$\epsilon_x = \frac{\partial u_x}{\partial x} \; ; \; \epsilon_y = \frac{\partial u_y}{\partial y} \; ; \; \epsilon_z = \frac{\partial \zeta}{\partial z}$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \; ; \; \gamma_{xz} = \gamma_{zx} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \; ; \; \gamma_{yz} = \gamma_{zy} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

Hooke's law for isotropic materials

Pure extension



Definition

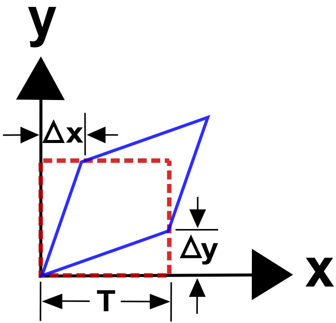
$$\epsilon = \frac{\text{extension}}{\text{original length}} = \frac{\Delta L}{L_o}$$

General Definition

$$\epsilon_x = \frac{\partial\, u_x}{\partial y} \qquad \epsilon_y = \frac{\partial\, u_y}{\partial y} \qquad \epsilon_z = \frac{\partial\, u_z}{\partial z}$$

Hooke's law for isotropic materials

Pure shear



General Definition

$$\tan(\gamma) \approx \gamma = \frac{\Delta x + \Delta y}{T}$$

General Definition

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$
$$\gamma_{xz} = \gamma_{zx} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$
$$\gamma_{yz} = \gamma_{zy} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

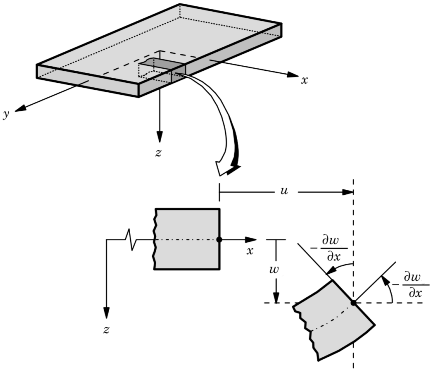
Bending waves in thin plates - Equations of motion

Assumptions of Kirchhoff for isotropic thin plates

(H1) Straight lines perpendicular to the mid-surface (i.e., transverse normals) before deformation remain straight after deformation.

(H2) The transverse normals do not experience elongation (i.e., they are in- extensible).

(H3) The transverse normals rotate such that they remain perpendicular to the middle surface after deformation.



Bending waves in thin plates - Equations of motion

- **Vertical displacement:**

(H1) $\Rightarrow \epsilon_z = \dfrac{\partial u_z}{\partial z} = 0 \Rightarrow u_z$ in independent of z

$$u_x(x,y,z,t) = w(x,y,t)$$

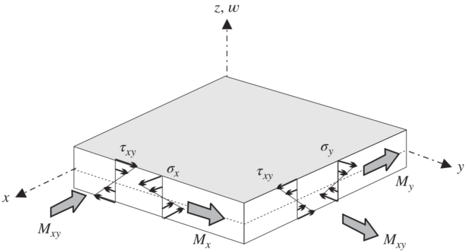
- **Horizontal displacements:**

(H3) $\Rightarrow \gamma_{xz} = \gamma_{zx} = \dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}$ and $\gamma_{yz} = \gamma_{zy} = \dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y}$

$$u_x(x,y,z,t) = -z\frac{\partial w(x,y,t)}{\partial x}$$

$$u_y(x,y,z,t) = -z\frac{\partial w(x,y,t)}{\partial y}$$

Bending waves in thin plates - Equations of motion



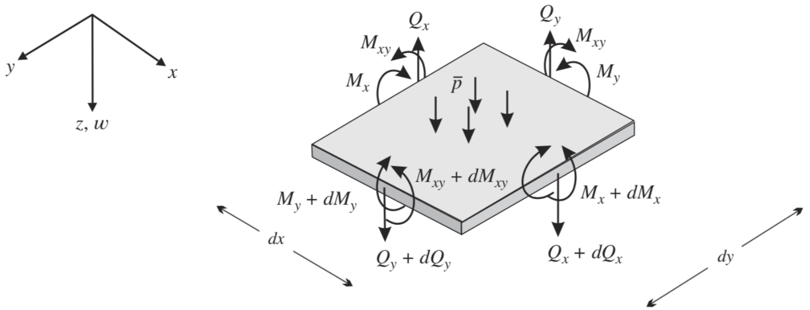
- **Expression of the moments per unit length:**

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z dz$$

$$M_{xy} = M_{yx} = \int_{-h/2}^{h/2} \tau_{xy} z dz$$

Bending waves in thin plates - Equations of motion



Bending waves in thin plates - Equations of motion

Newton's second law for the Moments

No rotation and shear neglected $\equiv$ **static**

⇒ **With respect to the $x-$axis:**

$$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}$$

⇒ **With respect to the $y-$axis:**

$$Q_x = \frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x}$$

Newton's second law for the Forces

Bending vibrations $w(x, y, t) \equiv$ **dynamic**

⇒ **With respect to the $z-$axis:**

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho h \frac{\partial^2 w}{\partial t^2}$$

Bending waves equation

$$D\nabla^4 w(x, y, t) + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0$$

D: bending stiffness

$$D = \frac{Eh^3}{12\,(1-\nu^2)}$$

h: thickness of the plate
ρ: density of the plate
E: Young modulus
ν: Poisson's ratio

Complex representation of harmonic waves:

1D example

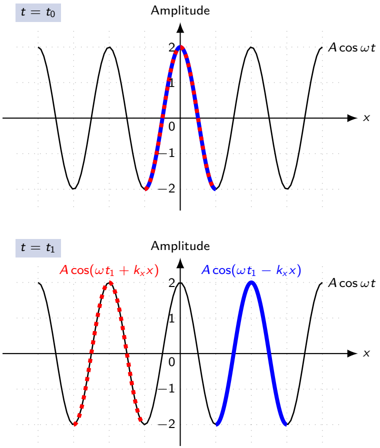
Propagative waves

⇒ **Complex representation:**

$$f_c(x) = \underbrace{Ae^{-jk_x x}}_{\text{outgoing wave}} + \underbrace{Be^{jk_x x}}_{\text{incoming wave}} \quad \text{with } k_x > 0$$

⇒ **Real representation:**

$$\begin{aligned} f_r(x,t) &= Re\left[f_c(x)e^{j\omega t}\right] \\ &= Re\left[\left(\underbrace{Ae^{-jk_x x}}_{\text{outgoing wave}} + \underbrace{Be^{jk_x x}}_{\text{incoming wave}}\right)e^{j\omega t}\right] \\ &= Re\left[Ae^{j(\omega t - k_x x)} + Be^{j(\omega t + k_x x)}\right] \\ &= \underbrace{A\cos(\omega t - k_x x)}_{\text{outgoing wave}} + \underbrace{B\cos(\omega t + k_x x)}_{\text{incoming wave}} \end{aligned}$$



Complex representation of harmonic waves:
1D example

Evanescent waves

⇒ **Complex representation:**

$$f_c(x) = Ae^{-k_x x} + Be^{k_x x} \text{ with } k_x > 0$$

if $x > 0 \Rightarrow f_c(x) = Ae^{-k_x x} + \cancel{Be^{k_x x}}$

if $x < 0 \Rightarrow f_c(x) = \cancel{Ae^{-k_x x}} + Be^{k_x x}$

⇒ **Real representation (with the convention $e^{j\omega t}$):**

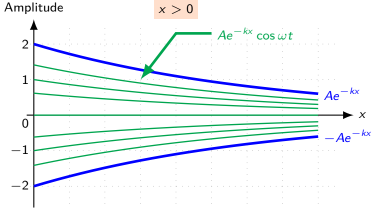
$$f_r(x,t) = Re\left[f_c(x)e^{j\omega t}\right]$$

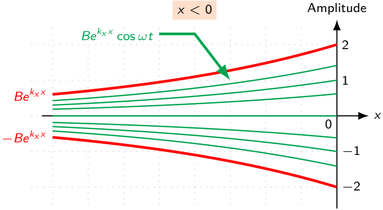
if $x > 0$

$$f_r(x,t) = Re\left[Ae^{-k_x x}e^{j\omega t}\right] = Ae^{-k_x x}\cos\omega t$$

if $x < 0$

$$f_r(x,t) = Re\left[Be^{k_x x}e^{j\omega t}\right] = Be^{k_x x}\cos\omega t$$





Bending waves in thin plates - Equations of motion

Homogeneous (In vacuum) equation of motion of an isotropic thin plate

$$D\nabla^4 w(x,y,t) + \rho h \frac{\partial^2 w(x,y,t)}{\partial t^2} = 0$$

Harmonic blending waves : $w(x,y,t) = w(x,y)e^{j\omega t} \Rightarrow \dfrac{\partial^2 w(x,y,t)}{\partial t^2} = -\omega^2 w(x,y)e^{j\omega t}$:

$$\nabla^4 w(x,y) - k_f^4 w(x,y) = 0$$

$\boldsymbol{k_f}$: **in vacuum wavenumber** $k_f^4 = \omega^2 \dfrac{\rho h}{D}$

Bilaplacian operator

$$\nabla^4 = \left(\nabla^2\right)^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

Bending waves in thin plates

General solution of homogeneous equation of motion

$$\nabla^4 w(x,y) - k_f^4 w(x,y) = 0 \Rightarrow \left(\nabla^4 - k_f^4\right) w(x,y) = \left(\nabla^2 + k_f^2\right)\left(\nabla^2 - k_f^2\right) w(x,y) = 0$$

$\Rightarrow$ **General solution**: $w(x,y) = w^+(x,y) + w^-(x,y)$

$$\underbrace{\left(\nabla^2 + k_f^2\right) w^+(x,y) = 0}_{\text{Progative wave equation}} \qquad \underbrace{\left(\nabla^2 - k_f^2\right) w^-(x,y) = 0}_{\text{Evanescent wave equation}}$$

$\Rightarrow$ Solutions found **using separation of variables method**:

$$w^+(x,y) = \left(A_x e^{-jk_x x} + B_x e^{jk_x x}\right)\left(A_y e^{-jk_y y} + B_y e^{jk_y y}\right)$$

$$w^-(x,y) = \left(A_x e^{-k_x x} + \cancel{B_x e^{k_x x}}\right)\left(A_y e^{-k_y y} + \cancel{B_y e^{k_y y}}\right)$$

Dispersion relation

$$k_f^2 = \omega\sqrt{\frac{\rho h}{D}} = k_x^2 + k_y^2$$

Bending waves in thin plates

Equation of motion of an isotropic thin plate in a **fluid**

$$\nabla^4 w(x,y) - k_f^4 w(x,y) = \frac{\overline{p}(x,y)}{D}$$

$\overline{p}(x,y)$: force density due to the dynamic fluctuation of the fluid

Effective wavenumber

⇒ **Pseudo-homogeneous** equation of motion

$$\nabla^4 w(x,y) - \gamma^4 w(x,y) = 0$$

⇒ γ : **effective wavenumber** (i.e. k_f modified by the presence of the heavy fluid)

$$\gamma^4 = \frac{\nabla^4 w(x,y)}{w(x,y)}$$

Chapter 2

Wave equation in plate and fluid

2. Acoustical waves in semi-infinite space

The acoustic wave equation

Assumptions to describe the propagation of waves in a fluid

(i) The fluid is an idealized **non viscous** fluid is initially assumed to be at rest

(ii) The ambient temperature T_0, pressure p_0, and density ρ_0 are constant with respect to time and space

(iii) A disturbance (or **perturbation**) causes a certain motion or waves in the fluid, which in turn causes **pressure fluctuations**

(iv) **Linear acoustic assumptions:**

- ⇒ All "acoustic" perturbations are supposed to be small with respect to the corresponding physical quantity
- ⇒ No static flow in the fluid

Pressure: $p_t(x, y, z, t) = p_0 + p(x, y, z, t)$ with $p(x, y, z, t) \ll p_0$

Density: $\rho_t(x, y, z, t) = \rho_0 + \rho(x, y, z, t)$ with $\rho(x, y, z, t) \ll \rho_0$

Flow: $\vec{u}_t(x, y, z, t) = \cancel{\vec{u}_0} + \vec{u}(x, y, z, t)$

Particle velocity: ($\neq$ sound velocity) $\vec{u}(x, y, z, t)$ with $|\vec{u}(x, y, z, t)| \sim p(x, y, z, t)$ and $\rho(x, y, z, t)$

The acoustic wave equation

The equation governing the propagation of waves in fluids is based on

1. The principle of **conservation of mass**

$$\frac{\partial \rho}{\partial t} + \rho_0 \overrightarrow{\nabla} \cdot \overrightarrow{u} = 0$$

2. The **Newton's second law** (the Euler equation)

$$\rho_0 \frac{\partial \overrightarrow{u}}{\partial t} + \overrightarrow{\nabla} p = 0$$

3. A **state equation** (thermodynamic: adiabatic transformation in a perfect gas)

$$p = c^2 \rho$$

Acoustic waves in semi-infinite space

Homogeneous acoustic wave equation

Combining the three previous equations we obtain the **wave equation**

$$\nabla^2 p(x, y, z, t) - \frac{1}{c^2} \frac{\partial^2 p(x, y, z, t)}{\partial t^2} = 0$$

$\boldsymbol{c}$: **sound velocity**

Acoustic waves in semi-infinite space

Homogeneous acoustic wave equation

Combining the three previous equations we obtain the **wave equation**

$$\nabla^2 p(x,y,z,t) - \frac{1}{c^2}\frac{\partial^2 p(x,y,z,t)}{\partial t^2} = 0$$

c: **sound velocity**

General solution of Helmholtz equation

Harmonic acoustic pressure : $p(x,y,z,t) = p(x,y,z)e^{j\omega t} \Rightarrow \frac{\partial^2 p(x,y,z,t)}{\partial t^2} = -\omega^2 p(x,y,z)e^{j\omega t}$:

$\Rightarrow$ **Helmholtz equation**: $\nabla^2 p(x,y,z) + k^2 p(x,y,z) = 0$

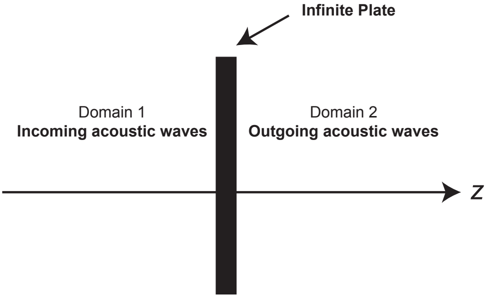
k: **acoustic wavenumber** $k^2 = \left(\frac{\omega}{c}\right)^2$

General solution of Helmholtz equation:

$$p(x,y,z) = \left(A_x e^{-jk_x x} + B_x e^{jk_x x}\right)\left(A_y e^{-jk_y y} + B_y e^{jk_y y}\right)\left(A_z e^{-jk_z z} + B_z e^{jk_z z}\right)$$

Acoustic waves in semi-infinite space

Only acoustic waves radiated by the plates are considered



Domain 1: Only incoming wave with respect to z

$$p(x,y,z) = \left(A_x e^{-jk_x x} + B_x e^{jk_x x}\right)\left(A_y e^{-jk_y y} + B_y e^{jk_y y}\right)\left(\cancel{A_z e^{-jk_z z}} + B_z e^{jk_z z}\right)$$

Domain 2: Only outgoing wave with respect to z

$$p(x,y,z) = \left(A_x e^{-jk_x x} + B_x e^{jk_x x}\right)\left(A_y e^{-jk_y y} + B_y e^{jk_y y}\right)\left(A_z e^{-jk_z z} + \cancel{B_z e^{jk_z z}}\right)$$

Acoustic intensity vector

The acoustic intensity vector is defined as

$$\overrightarrow{I} = \left\langle p_r \overrightarrow{u}_r \right\rangle_t$$

$\langle \cdot \rangle_t$: time average

p_r $\overrightarrow{u}_r$: **real** acoustic pressure and real particle velocity

Using complex representation for an harmonic acoustic wave

$$p_r = Re\left[p_c\right] = Re\left[P_c e^{j\omega t}\right] = \frac{P_c e^{j\omega t} + P_c^* e^{-j\omega t}}{2}$$

$$\overrightarrow{u}_r = Re\left[\overrightarrow{u}_c\right] = Re\left[\overrightarrow{U}_c e^{j\omega t}\right] = \frac{\overrightarrow{U}_c e^{j\omega t} + \overrightarrow{U}_c^* e^{-j\omega t}}{2}$$

we obtain

$$\overrightarrow{I} = \frac{1}{4}\left(P_c \overrightarrow{U}_c^* + P_c^* \overrightarrow{U}_c\right) = \frac{1}{2} Re\left[P_c \overrightarrow{U}_c^*\right] = \frac{1}{2} Re\left[P_c^* \overrightarrow{U}_c\right]$$

Remark: With the simplified notation used previously we have:

$$\overrightarrow{I} = \frac{1}{4}\left(p \overrightarrow{u}^* + p^* \overrightarrow{u}\right) = \frac{1}{2} Re\left[p \overrightarrow{u}^*\right] = \frac{1}{2} Re\left[p^* \overrightarrow{u}\right]$$

Chapter 3

Vibroacoustic coupling: case of an infinite plate

Chapter 3

Vibroacoustic coupling: case of an infinite plate

1. Vibroacoustic coupling: general view

Vibroacoustic coupling: general view

Continuity of the normal parietal velocity

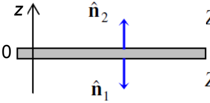
$$v(x, y) = u_n(x, y, 0)$$

- **Vibratory velocity** of the plate $v(x, y) = j\omega w(x, y)$
- **Particle velocity** of the acoustic wave: $\mathbf{u}(x, y, z) = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$, obtained using the **Euler equation**.
- **Euler equation** (Newton's second law in linear acoustic):

$$j\omega\rho_0\mathbf{u} + \nabla p = 0$$

$$v(x, y) = u_n(x, y, 0) \Rightarrow \left.\frac{\partial p(x, y, z)}{\partial z}\right|_{z=0} = \omega^2\rho_0 w(x, y)$$

Radiation impedance : $Z_r = \dfrac{\text{Parietal pressure}}{\text{Normal particle velocity}}$



$$Z_{r2} = \frac{p_2}{u_{n2}} = \frac{p_2}{u_{z2}} = \frac{p_2}{j\omega\, w}$$

$$Z_{r1} = \frac{p_1}{u_{n1}} = \frac{p_1}{-u_{z1}} = \frac{p_1}{-j\omega\, w}$$

Vibroacoustic coupling: general view

Equation of motion of an isotropic thin plate in a **fluid**

⇒ **Equation of motion:**
(without external mechanical excitation ≡ eigenvalue problem)

$$\begin{aligned} \nabla^4 w(x,y) - k_f^4 w(x,y) &= \frac{\overline{p}(x,y)}{D} \\ &= \frac{p_1(x,y,0) - p_2(x,y,0)}{D} \end{aligned}$$

⇒ **Effective wavenumber:**

$$\begin{aligned} \gamma^4 &= k_f^4 + \frac{p_1(x,y,0) - p_2(x,y,0)}{D\, w(x,y)} \\ &= k_f^4 \left(1 + \frac{p_1(x,y,0) - p_2(x,y,0)}{\rho h \omega^2 w(x,y)}\right) \\ &= k_f^4 \left(1 - j\frac{Z_{r1} + Z_{r2}}{\rho h \omega}\right) \end{aligned}$$

⇒ **Dispersion relation**

Pariatal pressures:

$p_1(x,y,0)$

$p_2(x,y,0)$

Radiation impedances:

$$Z_{r1} = -\frac{p_1(x,y,0)}{j\omega w(x,y)}$$

$$Z_{r2} = \frac{p_2(x,y,0)}{j\omega w(x,y)}$$

Bending stiffness:

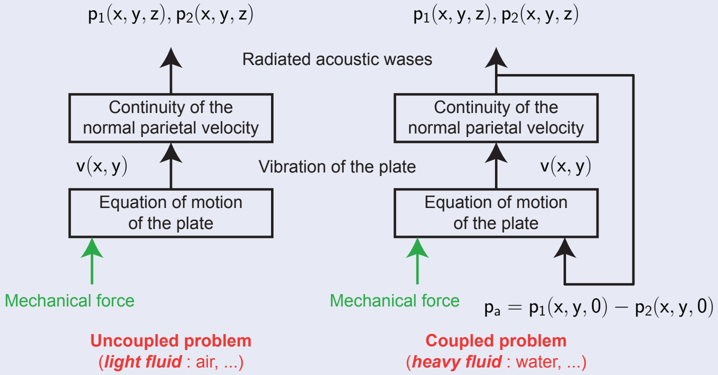
$$D = \omega^2 \frac{\rho h}{k_f^4}$$

Vibroacoustic coupling: general view

Coupled and **uncoupled** problem

Uncoupled problem: $Z_{r1}/\rho h\omega \ll 1$ and $Z_{r2}/\rho h\omega \ll 1 \Rightarrow \gamma \approx k_f$

Coupled problem: $Z_{r1}/\rho h\omega$ and $Z_{r2}/\rho h\omega$ cannot be neglected $\Rightarrow \gamma \neq k_f$



Vibroacoustic coupling: general view

Quantification of the vibroacoustic coupling

⇒ **Radiated acoustic intensity**:

$$\mathbf{I} \cdot \widehat{\mathbf{n}} = I_n = \frac{1}{2}\text{Re}\left\{p\, u_n^*\right\} = \frac{|u_n|^2}{2}\text{Re}\left\{Z_r\right\} = \frac{\omega^2}{2}\text{Re}\left\{Z_r\right\}|w|^2$$

⇒ **Acoustic power**:

$$P_a = \int_S \mathbf{I} \cdot \widehat{\mathbf{n}}\, dS = \frac{\omega^2}{2}\int_S \text{Re}\left\{Z_r\right\}|w|^2\, dS$$

⇒ **Radiation rate**:

$$\sigma = \frac{P_a}{\frac{\omega^2}{2}\int_S \rho_0 c |w|^2\, dS} = \frac{\int_S \text{Re}\left\{Z_r/\rho_0 c\right\}|w|^2\, dS}{\int_S |w|^2\, dS}$$

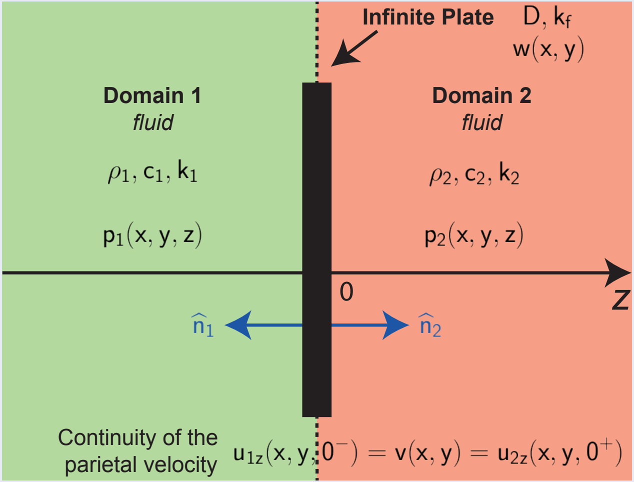
Chapter 3

Vibroacoustic coupling: case of an infinite plate

2. Acoustic radiation of the infinite plate

Vibroacoustic coupling: Case of an infinite plate

Diagram of the general problem



Vibroacoustic coupling: Case of an infinite plate

Equations of the problem

Domain 1: acoustic waves (Helmholtz equation)

$$\nabla^2 p_1(x,y,z) + k_1^2 p_1(x,y,z) = 0$$

Interface domain 1 / plate: continuity of parietal particle velocity and vibratory velocity of the plate

$$\left.\frac{\partial p_1(x,y,z)}{\partial z}\right|_{z=0} = u_{1z}(x,y,0) = \omega^2 \rho_1 w(x,y)$$

Plate: Equation of bending waves in a presence of the fluids

$$\nabla^4 w(x,y) - k_f^4 w(x,y) = \frac{p_1(x,y,0) - p_2(x,y,0)}{D} \quad \Leftrightarrow \quad \nabla^4 w(x,y) - \gamma^4 w(x,y) = 0$$

Interface domain 2 / plate: continuity of parietal particle velocity and vibratory velocity of the plate

$$\left.\frac{\partial p_2(x,y,z)}{\partial z}\right|_{z=0} = u_{2z}(x,y,0) = \omega^2 \rho_2 w(x,y)$$

Domain 2: acoustic waves (Helmholtz equation)

$$\nabla^2 p_2(x,y,z) + k_2^2 p_2(x,y,z) = 0$$

Vibroacoustic coupling: Case of an infinite plate

General solution of the problem

Acoustic pressure

Domain 1 (incoming waves): $p_1(x,y,z) = \left(A_{1x}e^{-jk_{1x}x} + B_{1x}e^{jk_{1x}x}\right)\left(A_{1y}e^{-jk_{1y}y} + B_{1y}e^{jk_{1y}y}\right)e^{jk_{1z}z}$

Domain 2 (outgoing waves): $p_2(x,y,z) = \left(A_{2x}e^{-jk_{2x}x} + B_{2x}e^{jk_{2x}x}\right)\left(A_{2y}e^{-jk_{2y}y} + B_{2y}e^{jk_{2y}y}\right)e^{-jk_{2z}z}$

Particle velocity

Domain 1: $u_{1z}(x,y,z) = \dfrac{j}{\omega\rho_1}\dfrac{\partial p_1(x,y,z)}{\partial z} = -\dfrac{k_{1z}}{\omega\rho_1}p_1(x,y,z)$

Domain 2: $u_{2z}(x,y,z) = \dfrac{j}{\omega\rho_2}\dfrac{\partial p_2(x,y,z)}{\partial z} = \dfrac{k_{2z}}{\omega\rho_2}p_2(x,y,z)$

Binding wave in the plate

Displacement: $w(x,y) = \left(W_x^A e^{-j\gamma_x x} + w_x^B e^{j\gamma_x x}\right)\left(W_y^A e^{-j\gamma_y y} + W_y^B e^{j\gamma_y y}\right)$

Velocity: $v(x,y) = j\omega w(x,y)$

Vibroacoustic coupling: Case of an infinite plate

General solution of the problem

Continuity of the velocities at $z = 0$

$$u_{1z}(x, y, 0^-) = v(x, y) = u_{2z}(x, y, 0^+)$$

$$\begin{aligned} & -\frac{k_{1z}}{\omega\rho_1}\left(A_{1x}e^{-jk_{1x}x} + B_{1x}e^{jk_{1x}x}\right)\left(A_{1y}e^{-jk_{1y}y} + B_{1y}e^{jk_{1y}y}\right) \\ = \quad & j\omega\left(W_x^A e^{-j\gamma_x x} + w_x^B e^{j\gamma_x x}\right)\left(W_y^A e^{-j\gamma_y y} + W_y^B e^{j\gamma_y y}\right) \\ = \quad & \frac{k_{2z}}{\omega\rho_2}\left(A_{2x}e^{-jk_{2x}x} + B_{2x}e^{jk_{2x}x}\right)\left(A_{2y}e^{-jk_{2y}y} + B_{2y}e^{jk_{2y}y}\right) \end{aligned}$$

These relations imply that:

$$k_{1x} = k_{2x} = \gamma_x \quad \text{and} \quad k_{1y} = k_{2y} = \gamma_y$$

Vibroacoustic coupling: Case of an infinite plate

General solution of the problem

The continuity relation can also take the form:

$$p_1(x,y,0) = -j\frac{\rho_1\omega^2}{k_{1z}}w(x,y) \qquad \text{and} \qquad p_2(x,y,0) = j\frac{\rho_2\omega^2}{k_{2z}}w(x,y)$$

The acoustic pressures may take the following form:

$p_1(x,y,z) = p_1(x,y,0)e^{jk_{1z}z}$

$p_2(x,y,z) = p_2(x,y,0)e^{-jk_{2z}z}$

with $k_{1z} = \sqrt{k_1^2 - \gamma^2}$ and $k_{2z} = \sqrt{k_2^2 - \gamma^2}$

Finally the acoustic pressure are written as follow:

$$\begin{cases} p_1(x,y,z) = -j\dfrac{\rho_1\omega^2}{\sqrt{k_1^2-\gamma^2}}w(x,y)e^{j\sqrt{k_1^2-\gamma^2}z} \\ p_2(x,y,z) = j\dfrac{\rho_2\omega^2}{\sqrt{k_2^2-\gamma^2}}w(x,y)e^{-j\sqrt{k_2^2-\gamma^2}z} \end{cases}$$

Radiation rate:

$$\sigma_i = \mathrm{Re}\left\{\frac{k_i}{k_{iz}}\right\} = \mathrm{Re}\left\{\frac{k_i}{\sqrt{k_i^2-\gamma^2}}\right\}$$

with $i = \{1, 2\}$

Tutored work

Vibroacoustic coupling: Case of an infinite plate

Solution of the dispersion equation

Dispersion relation:

$$\gamma^4 = k_f^4 \left(1 - j\frac{Z_{r1} + Z_{r2}}{\rho h \omega}\right)$$

Radiation impedances:

$$Z_{r1} = \frac{\rho_1 \omega}{k_{1z}} \quad \text{and} \quad Z_{r2} = \frac{\rho_2 \omega}{k_{2z}}$$

To simplify (two identical fluids) $\Rightarrow \rho_1 = \rho_2 = \rho_0$ and $k_1 = k_2 = k$

The dispersion relation becomes:

$$\frac{2\rho_0 k_f^4}{k_z} - j\rho h \left[\gamma^4 - k_f^4\right] = 0$$

with $k_z = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{k^2 - \gamma^2}$

Vibroacoustic coupling: Case of an infinite plate

Solution of the dispersion equation

Change of variable:

$$\kappa = \sqrt{\gamma^2 - k^2} = -jk_z \qquad \Rightarrow \qquad \gamma^2 = \kappa^2 + k^2$$

The acoustic pressures become:

$$\begin{cases} p_1(x,y,z) = -\dfrac{\rho_0 c\omega k}{\kappa} w(x,y) e^{-\kappa z} \\ p_2(x,y,z) = \dfrac{\rho_0 c\omega k}{\kappa} w(x,y) e^{\kappa z} \end{cases}$$

The dispersion relation becomes:

$$\frac{2\rho_0 k_f^4}{\rho h} + \kappa\left(\kappa^2 - k^2\right)^2 - \kappa k_f^4 = 0$$

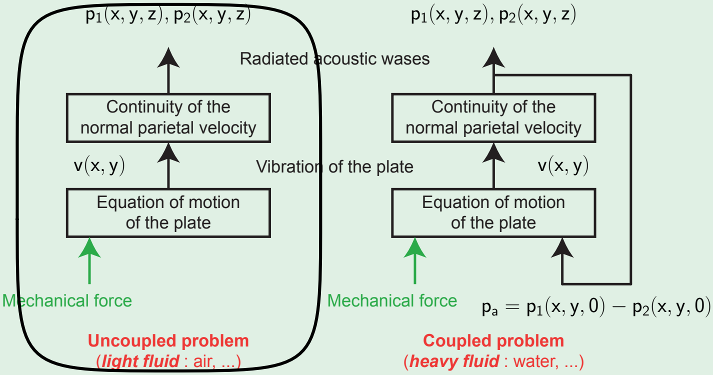
or

$$\kappa^5 + 2k^2\kappa^3 + \left[k^4 - k_f^4\right]\kappa + \frac{2\rho_0 k_f^4}{\rho h} = 0$$

$\Rightarrow$ **Five order polynomial equation**

Vibroacoustic coupling: Case of an infinite plate

Light fluids: **uncoupled** problem



Vibroacoustic coupling: Case of an infinite plate

Solution of the dispersion equation (Light fluid)

Light fluid $\Rightarrow \rho \gg \rho_0$: second order polynomial equation in κ^2:

$$\kappa^5 + 2k^2\kappa^3 + \left[k^4 - k_f^4\right]\kappa + \cancel{\frac{2\rho_0 k_f^4}{\rho h}} = 0$$

Two type of solutions:

First type of solution: $\kappa_1^2 = k_f^2 - k^2 \quad \Rightarrow \quad \kappa_1 = \pm\sqrt{k_f^2 - k^2} = \pm j\sqrt{k^2 - k_f^2}$

Second type of solution: $\kappa_2^2 = -\left(k^2 + k_f^2\right) \quad \Rightarrow \quad \kappa_2 = \pm j\sqrt{k^2 + k_f^2}$

Vibroacoustic coupling: Case of an infinite plate

Solution of the dispersion equation (Light fluid)

Light fluid $\Rightarrow \rho \gg \rho_0$: second order polynomial equation in κ^2:

$$\kappa^5 + 2k^2\kappa^3 + \left[k^4 - k_f^4\right]\kappa + \cancel{\frac{2\rho_0 k_f^4}{\rho h}} = 0$$

Two type of solutions:

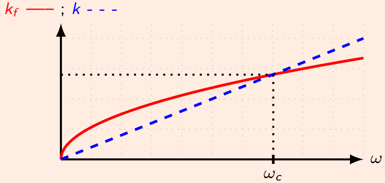
First type of solution: $\kappa_1^2 = k_f^2 - k^2 \quad \Rightarrow \quad \kappa_1 = \pm\sqrt{k_f^2 - k^2} = \pm j\sqrt{k^2 - k_f^2}$

Second type of solution: $\kappa_2^2 = -\left(k^2 + k_f^2\right) \quad \Rightarrow \quad \kappa_2 = \pm j\sqrt{k^2 + k_f^2}$

Critical frequency

$k = \dfrac{\omega}{c}$ and $k_f = \sqrt{\omega}\left(\dfrac{\rho h}{D}\right)^{1/4}$

$\omega_c = c^2\sqrt{\dfrac{\rho h}{D}}$ and $\dfrac{k_f^2}{k^2} = \dfrac{\omega_c}{\omega}$



Vibroacoustic coupling: Case of an infinite plate

Solution of the dispersion equation (Light fluid)

First type of solution

- $k > k_f \; (\omega > \omega_c)$
  $\Rightarrow$ **Propagative acoustic waves**

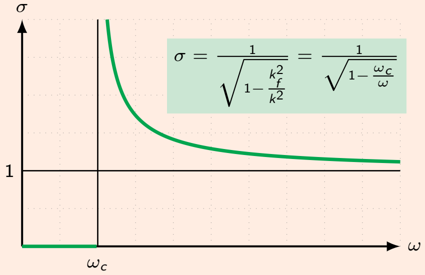
  $$p_2(x,y,z) = j\frac{\rho_0\omega^2}{\sqrt{k^2-k_f^2}}w(x,y)e^{-j\sqrt{k^2-k_f^2}z}$$

  $$\sigma = \frac{1}{\sqrt{1-\frac{k_f^2}{k^2}}} = \frac{1}{\sqrt{1-\frac{\omega_c}{\omega}}}$$

- $k < k_f \; (\omega < \omega_c)$
  $\Rightarrow$ **Evanescent acoustic waves**

  $$p_2(x,y,z) = -\frac{\rho_0\omega^2}{\sqrt{k_f^2-k^2}}w(x,y)e^{-\sqrt{k_f^2-k^2}z}$$

  $$\sigma = 0$$



- $\gamma = \pm k_f \Rightarrow$ **Propagative bending waves**

Vibroacoustic coupling: Case of an infinite plate

Solution of the dispersion equation (Light fluid)

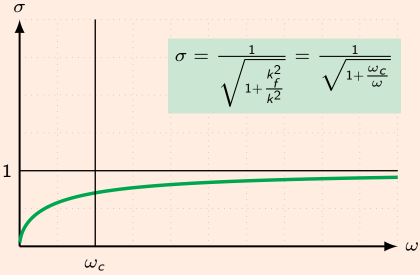
Second type of solution

- $-jk_z = -j\sqrt{k^2 + k_f^2}$
  $\Rightarrow$ **Propagative acoustic waves**

  $$p_2(x,y,z) = j\frac{\rho_0\omega^2}{\sqrt{k^2+k_f^2}}w(x,y)e^{-j\sqrt{k^2+k_f^2}z}$$

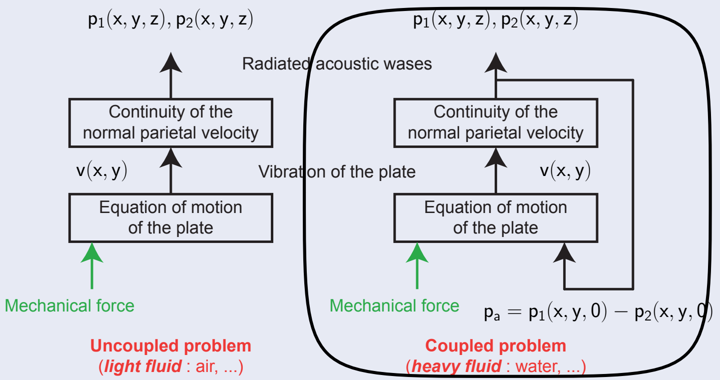
  $$\sigma = \frac{1}{\sqrt{1+\frac{k_f^2}{k^2}}} = \frac{1}{\sqrt{1+\frac{\omega_c}{\omega}}}$$

- $\gamma = \pm jk_f \Rightarrow$ **Evanescent bending waves**



Vibroacoustic coupling: Case of an infinite plate

Heavy fluids: **coupled** problem



Vibroacoustic coupling: Case of an infinite plate

General solutions for Heavy fluid

$$\kappa^5 + 2k^2\kappa^3 + \left[k^4 - k_f^4\right]\kappa + \frac{2\rho_0 k_f^4}{\rho h} = 0$$

Five order polynomial equation $\Rightarrow$ **5 roots** numerically found:

$\Rightarrow$ One negative real root: $\kappa = -\alpha_0$

$\Rightarrow$ Two complex conjugates roots: $\kappa = \alpha_1 \pm j\beta_1$ (they can become positive real roots if $\omega < \omega_c$)

$\Rightarrow$ Two complex conjugates roots: $\kappa = -\alpha_2 \pm j\beta_2$

with $\alpha_0, \alpha_1, \beta_1, \alpha_2, \beta_2 > 0$.

Considering **outgoing acoustic wave** which cannot grow to infinite, only solution with **negative** real and imaginary parts are kept:

$$\kappa = -\alpha_0 \quad \text{and} \quad \kappa = -\alpha_2 - j\beta_2$$

- $\kappa = -jk_z = -\alpha_0$; $p_2(x, y, z) = -\frac{\rho_0\omega^2}{\alpha_0} w(x, y) e^{-\alpha_0 z}$; $\gamma = \pm\sqrt{\alpha_0^2 + k^2}$
- $\kappa = -jk_z = -\alpha_2 - j\beta_2$; $p_2(x, y, z) = -\frac{\rho_0\omega^2}{\alpha_2 + j\beta_2} w(x, y) e^{-(\alpha_2 + j\beta_2)z}$; $\gamma = \pm\sqrt{(\alpha_2 + j\beta_2)^2 + k^2}$

Vibroacoustic coupling: Case of an infinite plate

Approximated solution for heavy fluid

Dispersion relation:

$$\gamma^4 = k_f^4 \left(1 - j\frac{Z_{r1} + Z_{r2}}{\rho h \omega}\right)$$

Radiation impedance: $Z_{r1} = Z_{r2} = \frac{\rho_0 \omega}{k_z} = \frac{\rho_0 \omega}{\sqrt{k^2 - \gamma^2}}$

$$\gamma^4 = k_f^4 \underbrace{\left(1 - j\frac{2\rho_0}{\rho h \sqrt{k^2 - \gamma^2}}\right)}_{\text{if } k > \gamma} = k_f^4 \underbrace{\left(1 + \frac{2\rho_0}{\rho h \sqrt{\gamma^2 - k^2}}\right)}_{\text{if } \gamma > k}$$

Vibroacoustic coupling: Case of an infinite plate

Approximated solution: $\gamma \approx k_f$ in the right-hand of dispersion relation

- if $k < \gamma$

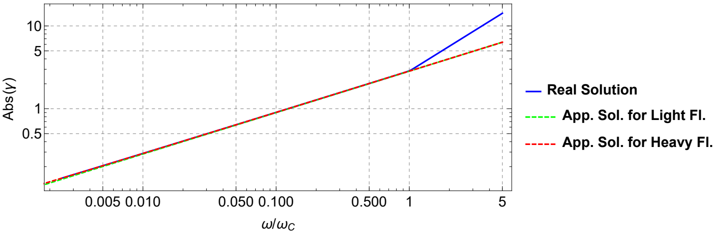
$$\gamma \approx \begin{Bmatrix} \pm 1 \\ \pm j \end{Bmatrix} k_f \left(1 + \frac{2\rho_0}{\rho h k_f \sqrt{1 - k^2/k_f^2}} \right)^{1/4} = \begin{Bmatrix} \pm 1 \\ \pm j \end{Bmatrix} k_f \left(1 + \frac{2\rho_0}{\rho h k_f \sqrt{1 - \omega/\omega_c}} \right)^{1/4}$$

- if $k > \gamma$

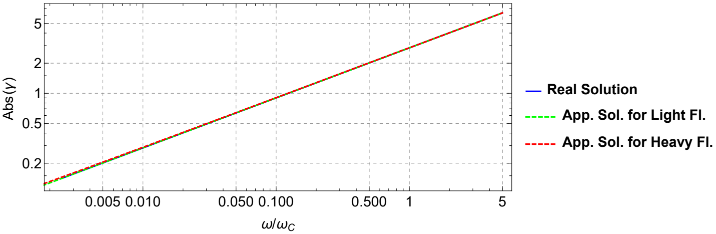
$$\gamma \approx \begin{Bmatrix} \pm 1 \\ \pm j \end{Bmatrix} k_f \left(1 - j \frac{2\rho_0}{\rho h k_f \sqrt{k^2/k_f^2 - 1}} \right)^{1/4} = \begin{Bmatrix} \pm 1 \\ \pm j \end{Bmatrix} k_f \left(1 - j \frac{2\rho_0}{\rho h k_f \sqrt{\omega/\omega_c - 1}} \right)^{1/4}$$

Comparison of solutions: Case of a light fluid (air)

Real solution: $\gamma = \pm\sqrt{\alpha_0^2 + k^2}$

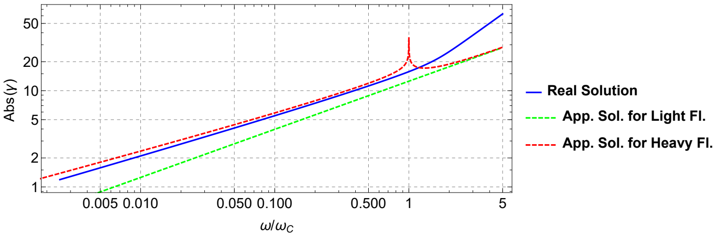


Real solution: $\gamma = \pm\sqrt{(\alpha_2 + j\beta_2)^2 + k^2}$

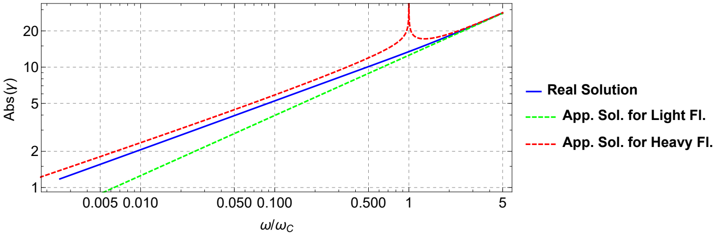


Comparison of solutions: Case of an heavy fluid (water)

Real solution: $\gamma = \pm\sqrt{\alpha_0^2 + k^2}$



Real solution: $\gamma = \pm\sqrt{(\alpha_2 + j\beta_2)^2 + k^2}$



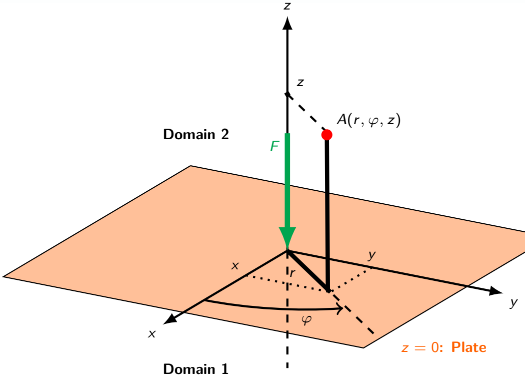
Chapter 3

Vibroacoustic coupling: case of an infinite plate

3. Radiation from infinite point-excited plates

Vibration of the infinite point-excited plate

⇒ **Problem description**



⇒ **Axisymmetric problem:** use of cylindrical coordinates (r, φ, z) (polar coordinates in the $(x0y)$-plan)

$$x = r\cos\varphi \quad \text{and} \quad y = r\sin\varphi$$

Vibration of the infinite point-excited plate

Equation of motion of an isotropic thin plate in a **fluid**

$$D\nabla^4 w(x,y) - \omega^2 \rho h w(x,y) = N_f(x,y) - p_a(x,y)$$

- $p_a(x,y) = -\overline{p}(x,y)$: force density due to the dynamic fluctuation of the fluid
  - $p_a(x,y) = 0$ for light fluid in both domains
  - $p_a(x,y) = p_2(x,y,0) - p_1(x,y,0)$ for heavy fluid in both domains
  - $p_a(x,y) = p_2(x,y,0)$ for a light fluid in domain 1 and an heavy fluid domain 2
- $N_f(x,y)$: force density due to the mechanical force which acts on the plate

  $\Rightarrow$ The force is supposed to be harmonic: $N_f(x,y,t) = N_f(x,y)e^{j\omega t}$

Vibration of the infinite point-excited plate

Equation of motion of an isotropic thin plate in a **fluid**

$$D\nabla^4 w(x,y) - \omega^2 \rho h w(x,y) = N_f(x,y) - p_a(x,y)$$

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  - $p_a(x,y) = 0$ for light fluid in both domains
  - $p_a(x,y) = p_2(x,y,0) - p_1(x,y,0)$ for heavy fluid in both domains
  - $p_a(x,y) = p_2(x,y,0)$ for a light fluid in domain 1 and an heavy fluid domain 2
- $N_f(x,y)$: force density due to the mechanical force which acts on the plate

  $\Rightarrow$ The force is supposed to be harmonic: $N_f(x,y,t) = N_f(x,y)e^{j\omega t}$

Case of an infinite point-excited plate

$$D\nabla^4 w(x,y) - \omega^2 \rho h w(x,y) = F\delta(x)\delta(y) - p_a(x,y)$$

- $\delta(x)$: Delta Dirac function
- F: Amplitude of the mechanical force

Vibration of the infinite point-excited plate

Bending wave equation in polar coordinates

⇒ Laplacian operator in polar coordinates

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2}$$

⇒ Axisymmetric problem ($\partial/\partial\varphi = 0$)

- The Laplacian operator becomes:

$$\nabla^2 f = \nabla_r^2 f = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) = \frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}$$

- The term $\delta(x)\delta(x)$ becomes $\dfrac{\delta(r)}{2\pi r}$

⇒ Bending wave equation in polar coordinates

$$D\nabla^4 w(r) - \omega^2 \rho h w(r) = F\frac{\delta(r)}{2\pi r} - p_a(r)$$

Solve for **light fluids** ($p_a(r) = 0$) using **Hankel transform**

Vibration of the infinite point-excited plate

The *Hankel transform*

⇒ Cartesian coordinates: **Fourier transform** → Polar coordinates: **Hankel transform**

⇒ **Mathematical definition:**

Direct: $HT\left[w(r)\right] = W\left(k_r\right) = \int_0^{\infty} w(r) J_0\left(k_r r\right) r dr$

Inverse: $HT^{-1}\left[W\left(k_r\right)\right] = w\left(r\right) = \int_0^{\infty} W\left(k_r\right) J_0\left(k_r r\right) k_r dk_r$

J_0: Bessel function of the first kind of order 0

Useful properties of *Hankel transform*

⇒ J_0 is by definition the solution of the **Helmholtz equation in polar coordinates in axisymmetric problems**

$$\nabla_r^2 J_0\left(k_r\right) + k_r^2 J_0\left(k_r\right) = 0$$

⇒ Direct and inverse **Hankel transform of the Laplacian**:

Direct: $-k_r^2 W\left(k_r\right) = \int_0^{\infty} \left[\nabla_r^2 w(r)\right] J_0\left(k_r r\right)$

Inverse: $\nabla_r^2 w\left(r\right) = \int_0^{\infty} \left[-k_r^2 W\left(k_r\right)\right] J_0\left(k_r r\right) k_r dk_r$ (cf. joined paper for the demonstration)

Vibration of the infinite point-excited plate

⇒ Since $HT\left[\frac{\delta(r)}{r}\right] = 1$, we have:

$$HT\left[D\nabla^4 w(r) - \omega^2 \rho h w(r) = F\frac{\delta(r)}{2\pi r} - p_a(r)\right] = Dk_r^4 \nabla^4 W(k_r) - \omega^2 \rho h W(k_r) = \frac{F}{2\pi} - P_a(k_r)$$

Vibration of the infinite point-excited plate

$\Rightarrow$ Since $HT\left[\dfrac{\delta(r)}{r}\right] = 1$, we have:

$$HT\left[D\nabla^4 w(r) - \omega^2 \rho h w(r) = F\frac{\delta(r)}{2\pi r} - p_a(r)\right] = Dk_r^4 \nabla^4 W(k_r) - \omega^2 \rho h W(k_r) = \frac{F}{2\pi} - P_a(k_r)$$

$\Rightarrow$ Light fluids are considered: $P_a(k_r) = 0$

Vibration of the infinite point-excited plate

⇒ Since $HT\left[\dfrac{\delta(r)}{r}\right] = 1$, we have:

$$HT\left[D\nabla^4 w(r) - \omega^2\rho h w(r) = F\frac{\delta(r)}{2\pi r} - p_a(r)\right] = Dk_r^4\nabla^4 W(k_r) - \omega^2\rho h W(k_r) = \frac{F}{2\pi} - P_a(k_r)$$

⇒ Light fluids are considered: $P_a(k_r) = 0$

⇒ The equation to solve is:

$$Dk_r^4\nabla^4 W(k_r) - \omega^2\rho h W(k_r) = \frac{F}{2\pi} = 0$$

Vibration of the infinite point-excited plate

⇒ Since $HT\left[\frac{\delta(r)}{r}\right] = 1$, we have:

$$HT\left[D\nabla^4 w(r) - \omega^2 \rho h w(r) = F\frac{\delta(r)}{2\pi r} - p_a(r)\right] = Dk_r^4\nabla^4 W(k_r) - \omega^2\rho h W(k_r) = \frac{F}{2\pi} - P_a(k_r)$$

⇒ Light fluids are considered: $P_a(k_r) = 0$

⇒ The equation to solve is:

$$Dk_r^4\nabla^4 W(k_r) - \omega^2\rho h W(k_r) = \frac{F}{2\pi} = 0$$

⇒ The Hankel transform of the displacement $w(r)$ is:

$$W(k_r) = \frac{F}{2\pi D\left(k_r^4 - k_f^4\right)}$$

Vibration of the infinite point-excited plate

⇒ Since $HT\left[\frac{\delta(r)}{r}\right] = 1$, we have:

$$HT\left[D\nabla^4 w(r) - \omega^2\rho h w(r) = F\frac{\delta(r)}{2\pi r} - p_a(r)\right] = Dk_r^4\nabla^4 W(k_r) - \omega^2\rho h W(k_r) = \frac{F}{2\pi} - P_a(k_r)$$

⇒ Light fluids are considered: $P_a(k_r) = 0$

⇒ The equation to solve is:

$$Dk_r^4\nabla^4 W(k_r) - \omega^2\rho h W(k_r) = \frac{F}{2\pi} = 0$$

⇒ The Hankel transform of the displacement $w(r)$ is:

$$W(k_r) = \frac{F}{2\pi D\left(k_r^4 - k_f^4\right)}$$

⇒ **Difficulty:** compute $w(r) = HT^{-1}\left[W(k_r)\right]$

Vibration of the infinite point-excited plate

⇒ **Calculation of inverse Hankel transform:**

$$w(r) = HT^{-1}\left[W\left(k_r\right)\right] = \int_0^{\infty} \frac{F}{2\pi D\left(k_r^4 - k_f^4\right)} J_0\left(k_r r\right) k_r dk_r$$

⇒ **Steps of calculation:**

Step 1 Transform the integral from $\int_0^{\infty}$ to $\int_{-\infty}^{+\infty}$

$$J_0(\alpha) = \frac{1}{2}\left[H_0^{(1)}(\alpha) - H_0^{(1)}(-\alpha)\right] \rightarrow w(r) = \frac{F}{4\pi D}\int_{-\infty}^{+\infty} \underbrace{\frac{H_0^{(1)}\left(k_r r\right)}{k_r^4 - k_f^4}}_{f(k_r)} k_r dk_r$$

$H_0^{(1)}$: Hankel function of the first kind

Step 2 Find de singularities of $f\left(k_r\right)$

Step 3 Express the integral as a part of an integral in the complex plan

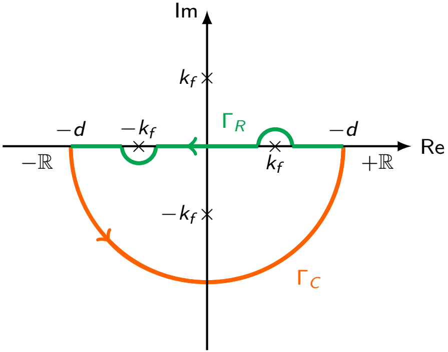
Step 4 Use the ***Residue Theorem*** and **second Jordan's lemma**

Vibration of the infinite point-excited plate

Step 2

The function $f\left(k_r\right)$ diverges if $k_r^4 - k_f^4 = 0$, the **singularities** are: $k_r = \pm k_f$ and $k_r = \pm j k_f$

Step 3



$$\oint_\Gamma f(z)dz = \int_{\Gamma_R} f(x)dx + \int_{\Gamma_C} f(z)dz$$

$$\int_{-\infty}^{+\infty} f(x)dx = - \lim_{d\to\infty} \int_{\Gamma_R} f(x)dx$$

$$= - \lim_{d\to\infty} \left[\oint_\Gamma f(z)d - \int_{\Gamma_C} f(z)dz \right]$$

$$= - \oint_\Gamma f(z)d + \int_{\Gamma_C} f(z)dz$$

$\Rightarrow \oint_\Gamma f(z)dz$ calculated using **residue theorem**

$\Rightarrow \int_{\Gamma_C} f(z)dz = 0$ (**second Jordan's lemma**)

Vibration of the infinite point-excited plate

Step 4

Residue theorem

We consider a complex-valued function f, the residue theorem states:

$$\oint_{\Gamma} f(z)dz = 2\pi j \sum_{n=1}^{N_s} I\left(\Gamma, a_n\right) \mathrm{Res}\left(f, a_n\right)$$

- N_s: number of singularities inside Γ
  $\Rightarrow$ 2 in the our problem: $a_1 = k_f$ and $a_2 = -jk_f$
- $\mathrm{Res}\left(f, a_n\right)$: residue of f at a_n:

$$\mathrm{Res}\left(f, a_n\right) = \lim_{z \to a_n} \left(z - a_n\right) f(z)$$

  $\Rightarrow$ In the our problem: $\mathrm{Res}\left(f, k_f\right) = \frac{H_0^{(1)}\left(k_f r\right)}{4k_f^2}$ and $\mathrm{Res}\left(f, -jk_f\right) = -\frac{H_0^{(1)}\left(-jk_f r\right)}{4k_f^2}$
- $I\left(\Gamma, a_n\right)$: winding number of the curve Γ about the point a_n
  $\Rightarrow$ If Γ is a positively oriented simple closed curve: $I\left(\Gamma, k_f\right) = I\left(\Gamma, -jk_f\right) = 1$

Vibration of the infinite point-excited plate

Final expression of the displacement $w(r)$

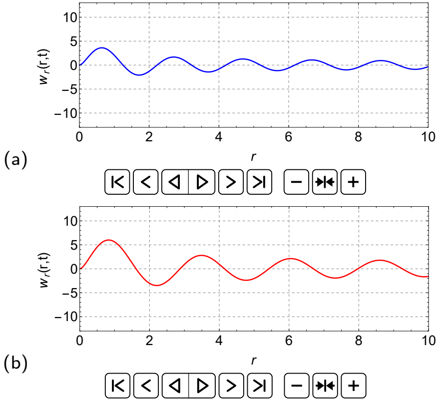
$$w(r) = -j \frac{F}{8Dk_f^2} \left[H_0^{(1)}(k_f r) - H_0^{(1)}(-jk_f r) \right]$$

Plot of the real displacement

$$w_r(r,t) = \mathrm{Re}\left[w(r) e^{j\omega t} \right]$$

(a) $\omega > \omega_c$

(b) $\omega < \omega_c$



Radiation from infinite point-excited plates:

Case of light fluid

First type of solution for $k > k_f$ $(\omega > \omega_c)$

⇒ **Propagative acoustic waves**

$$p_2(x,y,z) = j\frac{\rho_0\omega^2}{\sqrt{k^2-k_f^2}}w(x,y)e^{-j\sqrt{k^2-k_f^2}z}$$

$$p_2(r,z) = j\frac{\rho_0\omega^2}{\sqrt{k^2-k_f^2}}w(r)e^{-j\sqrt{k^2-k_f^2}z}$$

$$p_2(r,z) = \frac{F\rho_0\omega^2}{8Dk_f^2\sqrt{k^2-k_f^2}}\left[H_0^{(1)}(k_f r) - H_0^{(1)}(-jk_f r)\right]e^{-j\sqrt{k^2-k_f^2}z}$$