Slow-Fast analysis of a resonant oscillator coupled to a Bi-Stable Nonlinear Energy Sink (BNES) and a Piezoelectric Harvester

EURODINAME

9 juin 2026

Audrey Couineaux,

B. Bergeot, S. Berger, M.-L. Gobert, V. Denis, G. Poulin Vittrant, M. Bavencoffe

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES CENTRE VAL DE LOIRE

LaMé Laboratoire de Mécanique Gabriel Lamé

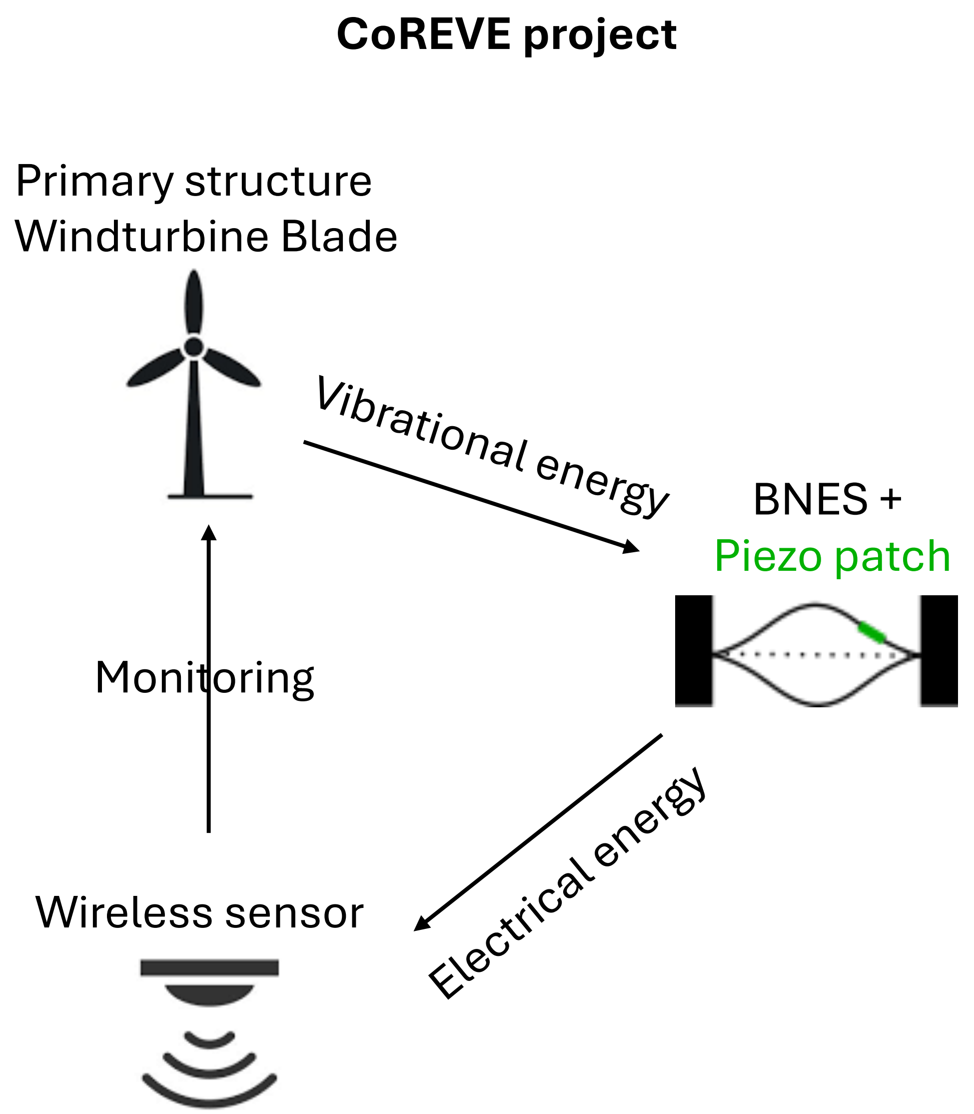
GREMAN matériaux microélectronique acoustique nanotechnologies

Context

https://www.youtube.com/watch?v=kSsnHSvMtpw

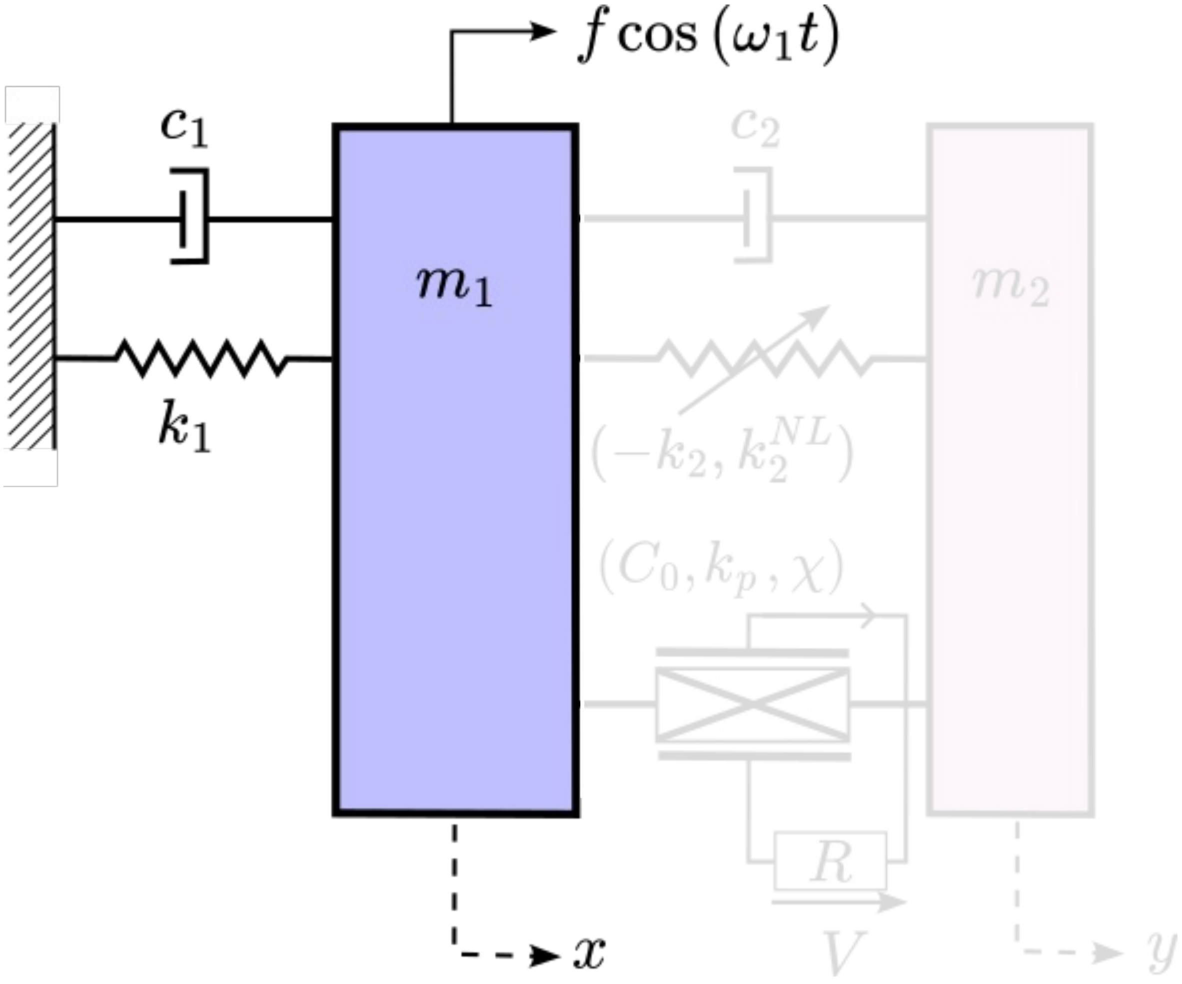
- **Wind turbine flutter**
- **CoREVE project:**
  - ***Co****ntrôle et* ***R****écupération d'****E****nergie* ***V****ibratoire dans les* ***E****oliennes*
  - Funded by **Région Centre Val de Loire**
- **2 objectives**
  - Vibration mitigation
  - Energy harvesting for wireless monitoring

Context



- **Wind turbine flutter**
- **CoREVE project:**
  - ***Co****ntrôle et* ***R****écupération d'****E****nergie* ***V****ibratoire dans les* ***E****oliennes*
  - Funded by **Région Centre Val de Loire**
- **2 objectives**
  - Vibration mitigation
  - Energy harvesting for wireless monitoring

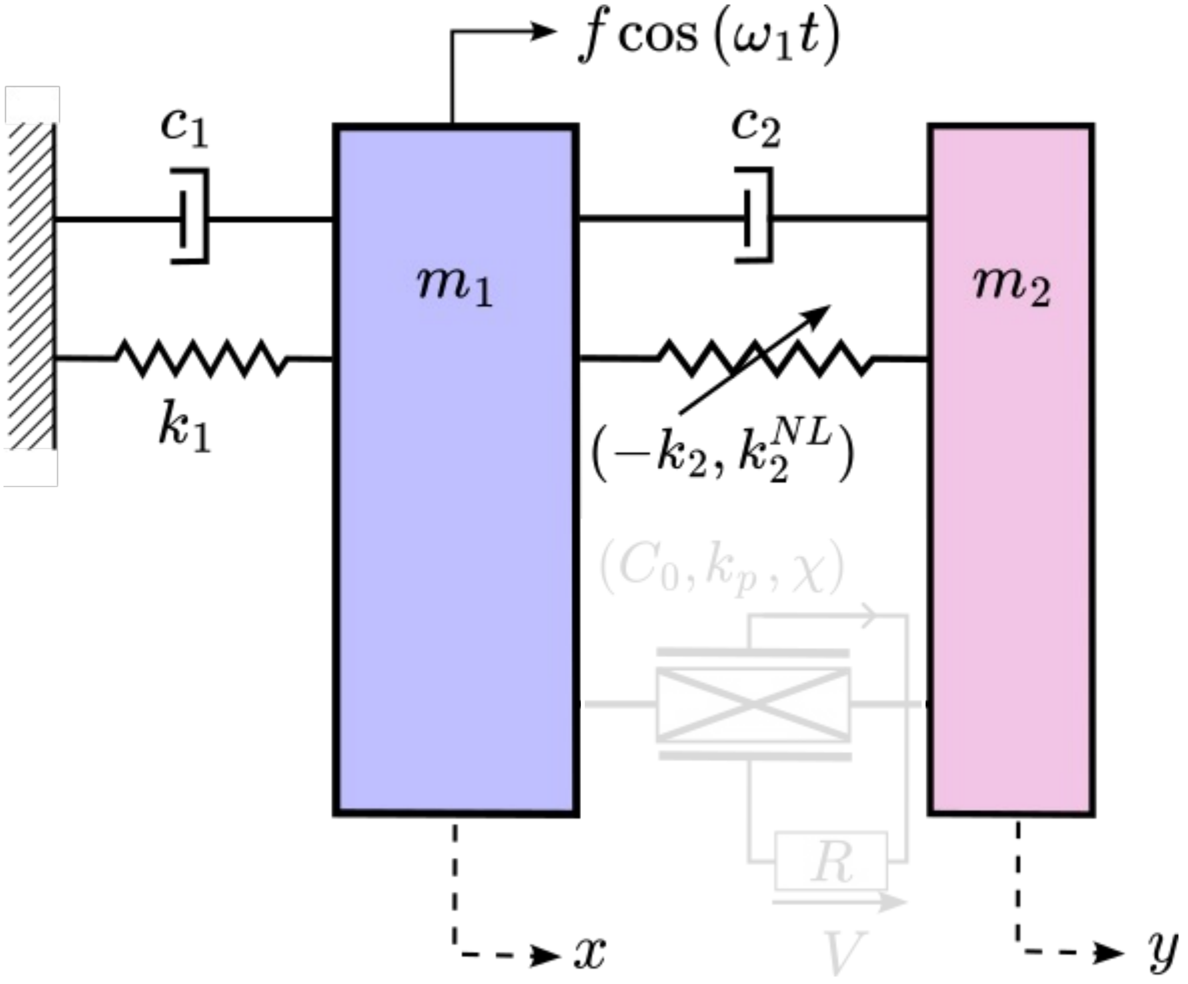
Model description



Academic model

- **Primary system**
  - Linear, resonant
  - Harmonic excitation

Model description

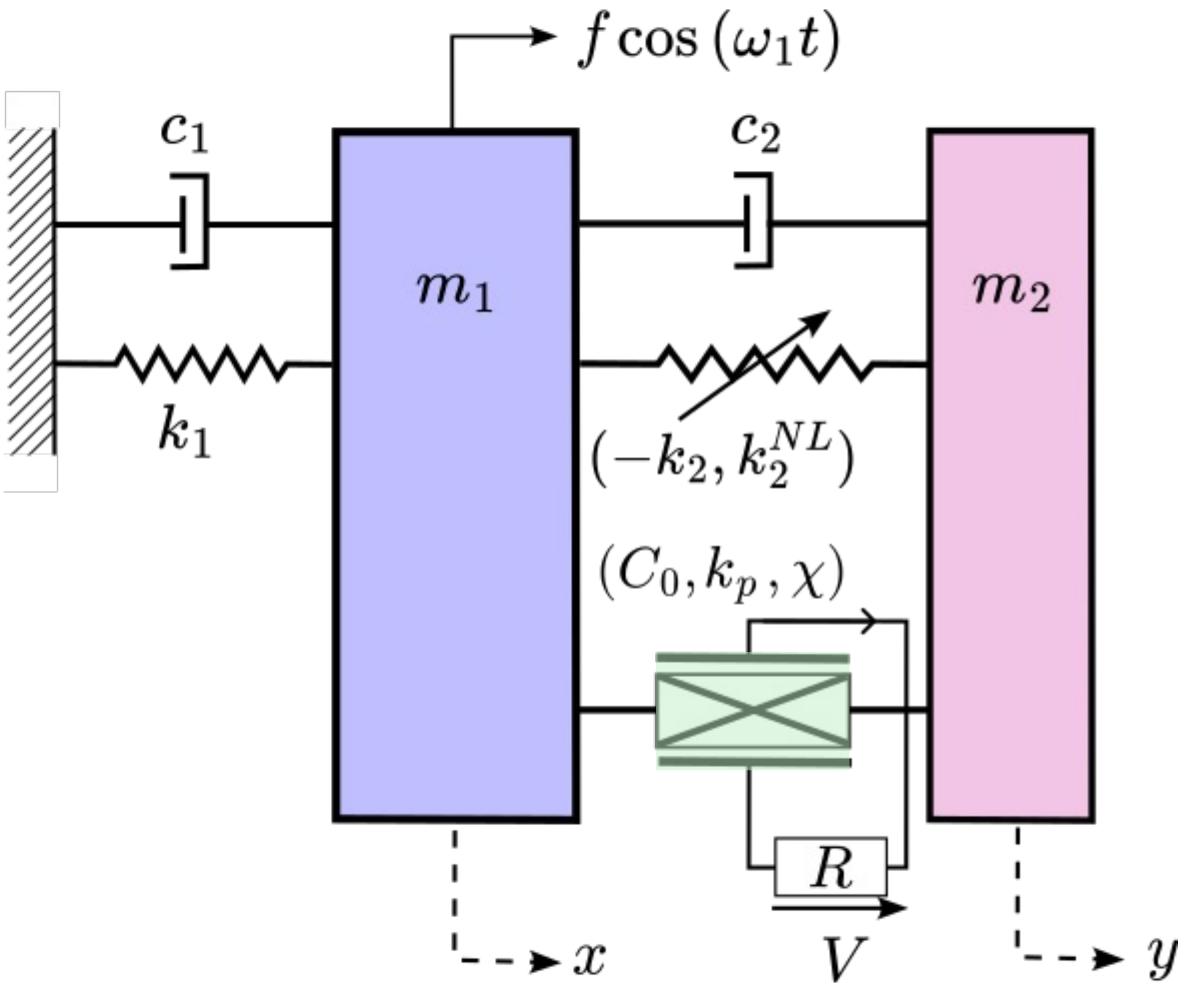


Academic model

- **Primary system**
  - Linear, resonant
  - Harmonic excitation
- **BNES**[1] **damper** (Bistable Nonlinear Energy Sink)
  - Nonlinear cubic stiffness k_2^{NL}
  - Linear negative stiffness $-k_2$ (bi-stability**)**
  - 3 equilibrium positions
    - 1 instable $y = 0$
    - 2 stables $y = \pm y_e$

[1] [Manevitch et al. (2013), J. Appl. Mech. 81(4)]

Model description

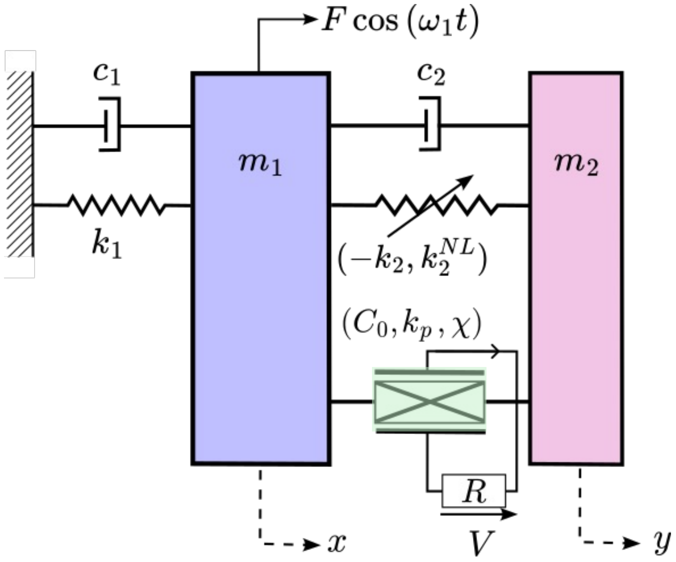


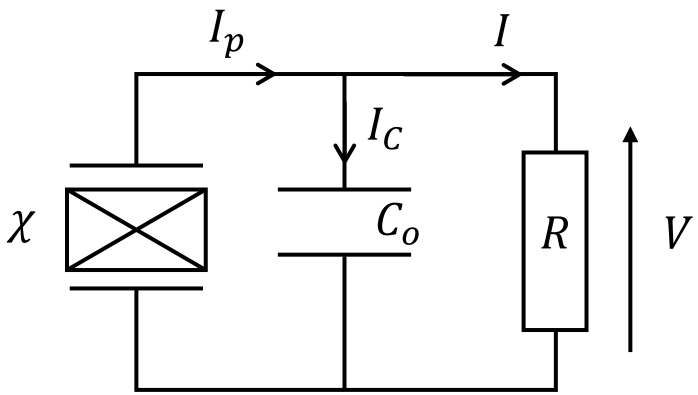
Academic model

- **Primary system**
  - Linear, resonant
  - Harmonic excitation
- **BNES[1] damper** (Bistable Nonlinear Energy Sink)
  - Nonlinear cubic stiffness k_2^{NL}
  - Linear negative stiffness $-k_2$ (bi-stability)
  - 3 equilibrium positions
    - 1 instable $y = 0$
    - 2 stables $y = \pm\, y_e$
- **Piezoelectric harvester**
  - Generator convention
  - Simple electrical circuit (singular resistor)

[1][Manevitch et al. (2013), J. Appl. Mech. 81(4)]

Model equations





Mechanical equations of motion:

$$m_1\, x'' + c_1 x' + k_1 x + c_2(x' - y') - (k_2 - k_p)(x - y) + k_{2NL}(x - y)^3 - \chi V = F\cos(\omega_1 t)$$

$$m_2\, y'' + c_2(y' - x') - (k_2 - k_p)(y - x) + k_{2NL}(y - x)^3 + \chi V = 0$$

$k_2 - k_p > 0$ **to conserve a BNES**

Electrical circuit equation:

$$C_0 V' + \frac{1}{R} V = \chi(y' - x')$$

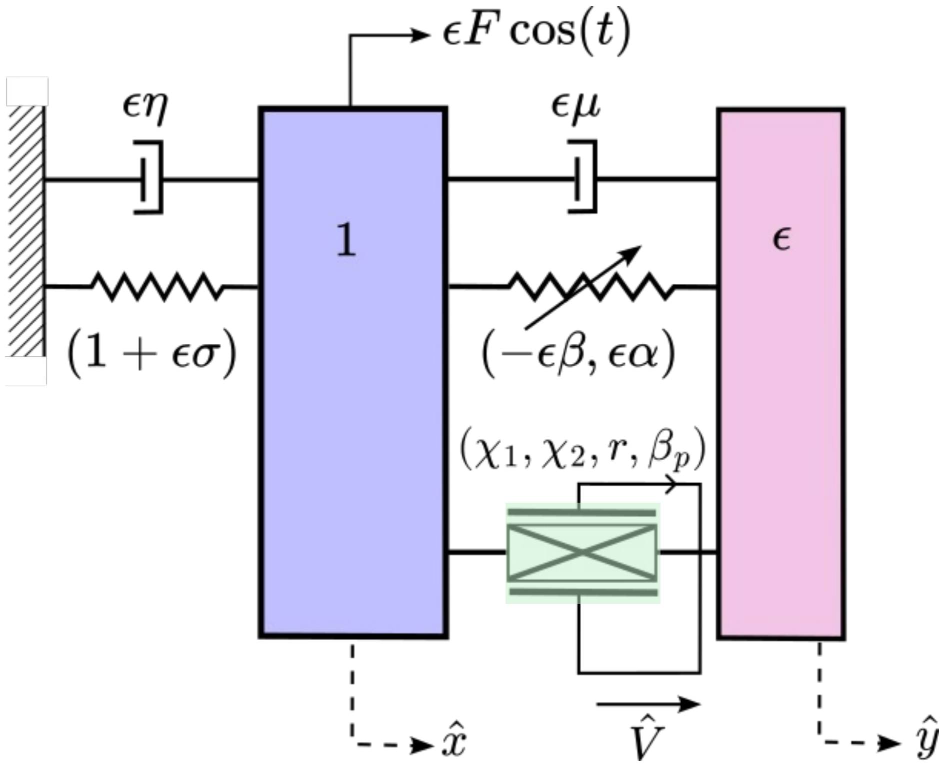
Model equations

Dimensionless equations and parameters

Rescaled quantities:

$$\hat{t} = t\omega_1; \hat{x} = \frac{x}{L}; \hat{y} = \frac{y}{L}; \hat{V} = \frac{V}{u}$$

$\epsilon = \frac{m_2}{m_1} \ll 1$: mass ratio between BNES and PS



Mechanical equations of motion:

$$\hat{x}'' + \epsilon\eta\hat{x}' + (1+\epsilon\sigma)\hat{x} + \epsilon\mu(\hat{x}' - \hat{y}') - \epsilon(\beta - \beta_p)(\hat{x} - \hat{y}) + \epsilon\alpha(\hat{x} - \hat{y})^3 - \epsilon\chi_1\hat{V} = \epsilon F\cos(\hat{t})$$

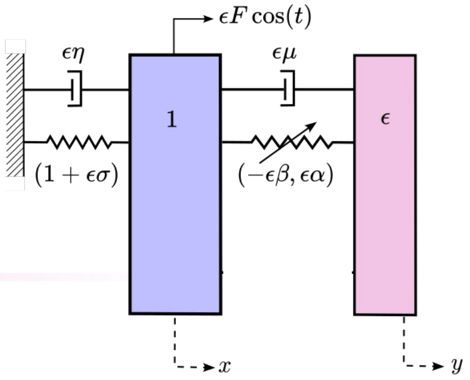
$$\epsilon\hat{y}'' + \epsilon\mu(\hat{y}' - \hat{x}') - \epsilon(\beta - \beta_p)(\hat{y} - \hat{x}) + \epsilon\alpha(\hat{y} - \hat{x})^3 + \epsilon\chi_1\hat{V} = 0$$

Electrical circuit equation:

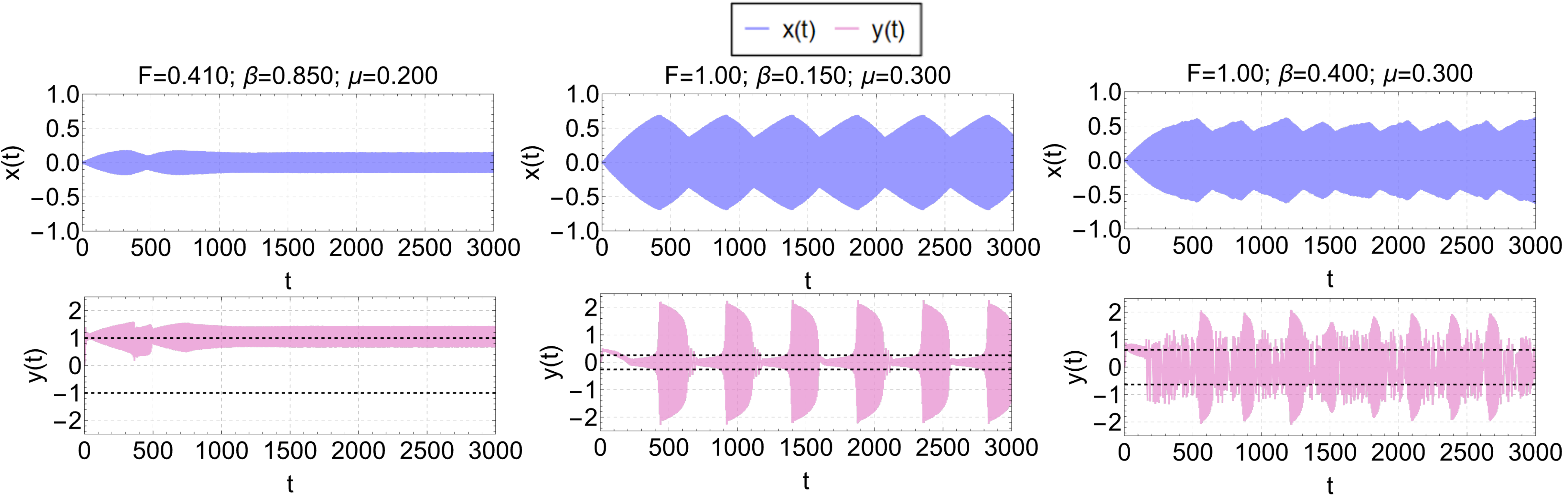
$$\hat{V}' + r\hat{V} - \chi_2(\hat{y}' - \hat{x}') = 0$$

Diversity of vibrational regimes

Without *piezoelectric element*



$\epsilon = 0{,}005$; $\eta = 0{,}5$; $\sigma = 0$; $\alpha = 0{,}75$

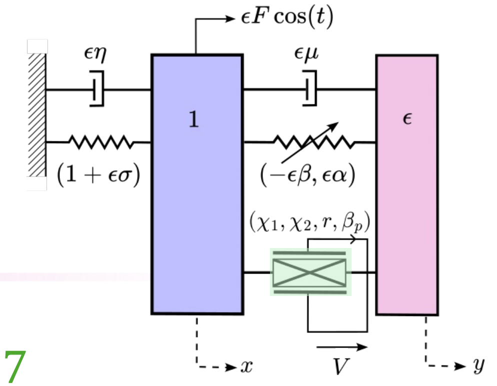


Regimes:

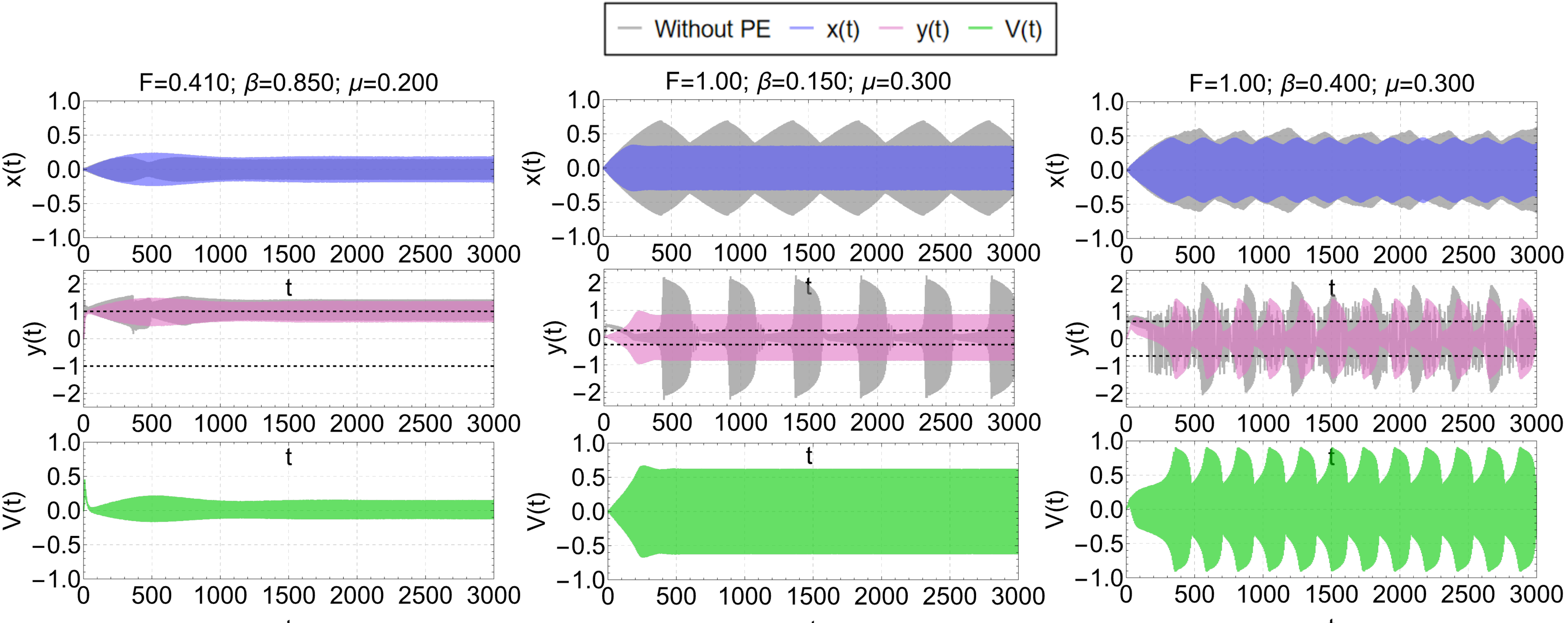
- Periodic
- Strongly modulated (SMR)
- Chaotic
- Zero centered
- Non-zero equilibria position centered

Diversity of vibrational regimes

With piezoelectric element



$\epsilon = 0{,}005$; $\eta = 0{,}5$; $\sigma = 0$; $\alpha = 0{,}75$; $\beta_p = 0{,}1$; $r = 0{,}1$; $\chi_1 = 0{,}7$; $\chi_2 = 0{,}7$



Methodology & Representation

Fast-Slow analysis

Variable substitution

$z = x - y$

1:1 resonance capture assumption

x, z, V amplitude-phase modulation:

$$x(\mathrm{t}) = \boldsymbol{a}(\boldsymbol{t})\ \cos(t + \boldsymbol{\theta}(\boldsymbol{t}))$$
$$z(\mathrm{t}) = \boldsymbol{b}(\boldsymbol{t}) + \boldsymbol{c}(\boldsymbol{t}) \cos\big(t + \boldsymbol{\varphi}(\boldsymbol{t})\big)$$
$$V(\mathrm{t}) = function[c(t), \varphi(t)]$$

Derivation of the **modulation flow** using **MSHBM**[2,3] (Multiple Scale/ Harmonic Balance Method) :

$$\dot{a} = \epsilon f_1(a, b, c, \theta, \varphi)$$
$$\dot{\theta} = \epsilon f_2(a, b, c, \theta, \varphi)$$
$$\dot{b} = g_1(a, b, c, \theta, \varphi, \epsilon)$$
$$\dot{c} = g_2(a, b, c, \theta, \varphi, \epsilon)$$
$$\dot{\varphi} = g_3(a, b, c, \theta, \varphi, \epsilon)$$

[2]MSBMD: [Luongo & Zulli (2012), Nonlinear Dyn, 70(3)]
[3]Application to BNES: [Bergeot & Berger (2024), Physica D, 460]

Methodology & Representation

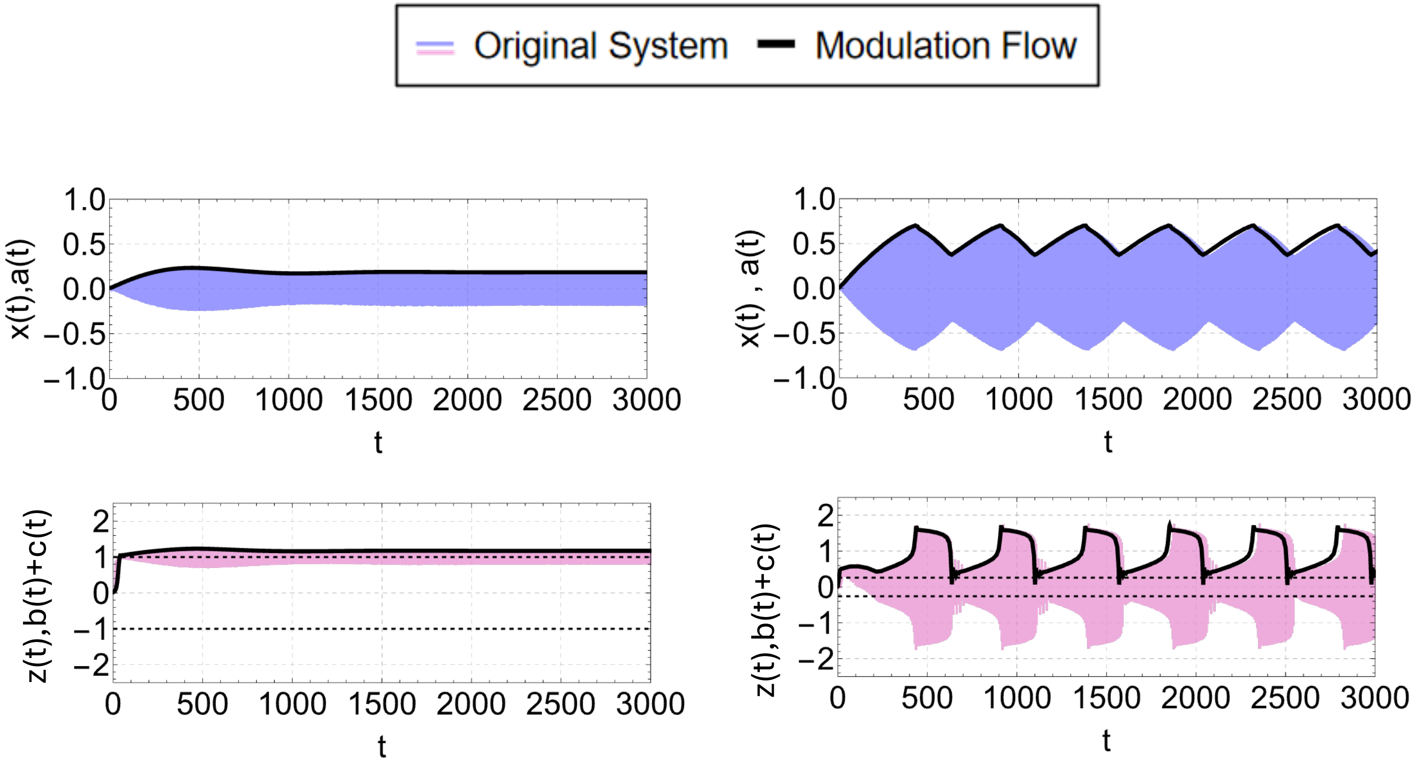
Fast-Slow system

- **Simpler system behavior**

- **Presence of $\epsilon \ll 1 \Rightarrow$ Fast-Slow system**

Derivation of the modulation flow using **MSHBM**[2,3]:

$$\dot{a} = \epsilon f_1(a,b,c,\theta,\varphi)$$
$$\dot{\theta} = \epsilon f_2(a,b,c,\theta,\varphi)$$
$$\dot{b} = g_1(a,b,c,\theta,\varphi,\epsilon)$$
$$\dot{c} = g_2(a,b,c,\theta,\varphi,\epsilon)$$
$$\dot{\varphi} = g_3(a,b,c,\theta,\varphi,\epsilon)$$



Original System: Periodic
Modulation Flow: Equilibrium

Original System: SMR
Modulation Flow: Periodic

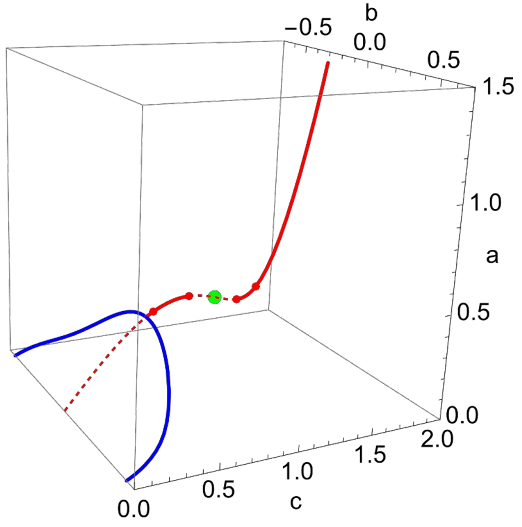
[2]MSBMD: [Luongo & Zulli (2012), Nonlinear Dyn, 70(3)]
[3]Application to BNES: [Bergeot & Berger (2024), Physica D, 460]

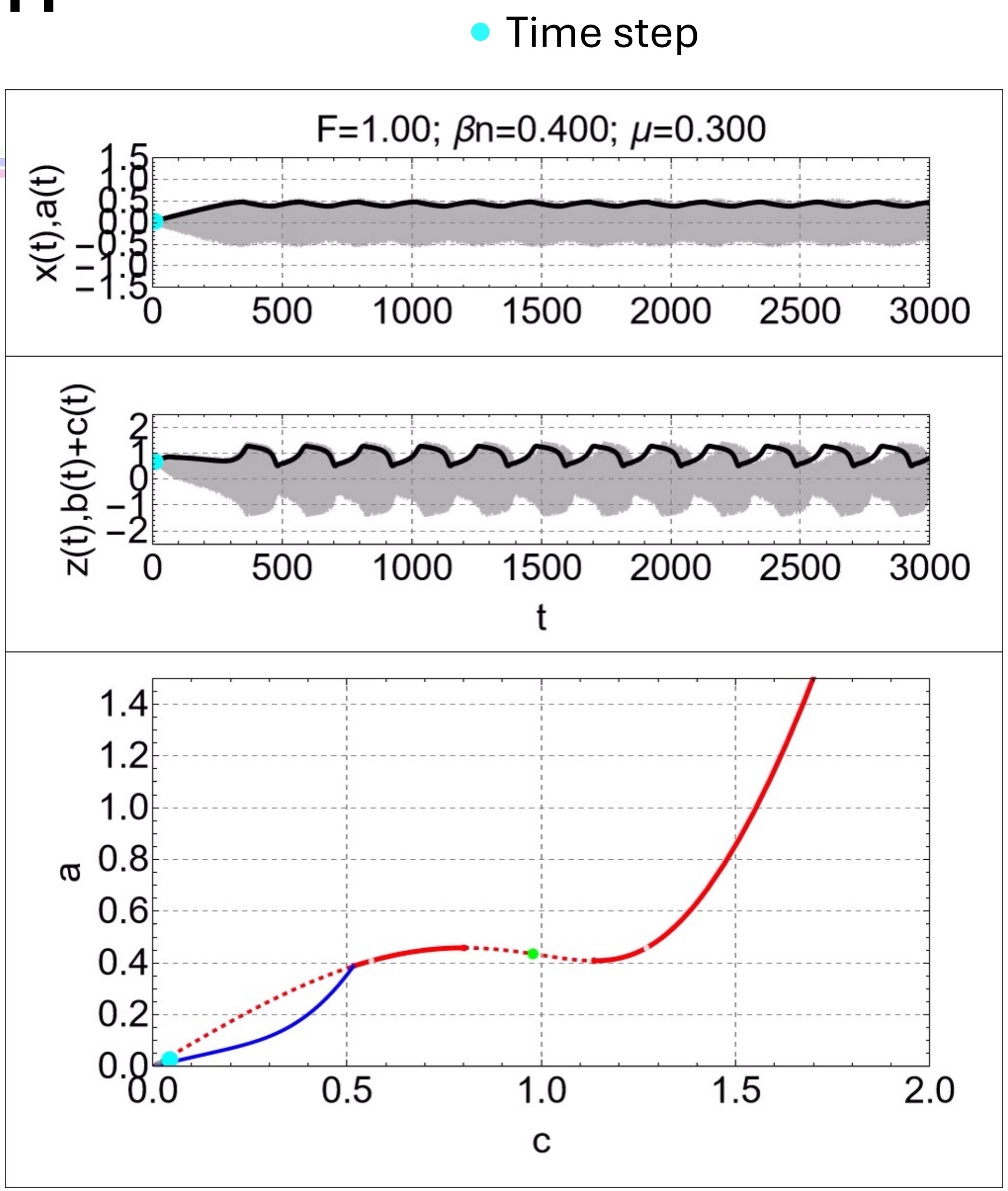
Methodology & Representation

Critical Manifold $\mathcal{M}_0$

Critical Manifold:

- $\mathcal{M}_0 = \left\{ (a,b,c,\theta,\varphi) \,\middle|\, \begin{array}{r} g_1(a,b,c,\theta,\varphi,0) = 0 \\ g_2(a,b,c,\theta,\varphi,0) = 0 \\ g_3(a,b,c,\theta,\varphi,0) = 0 \end{array} \right\}$
- 2 branches:
  - $b = 0$ (red)
  - $b \neq 0$ (blue), symmetric with respect to the plane $b = 0$



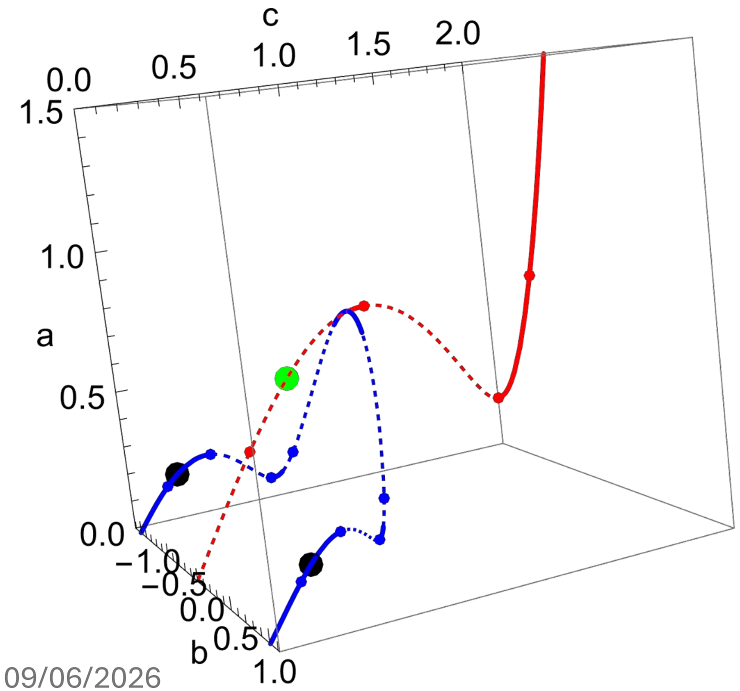


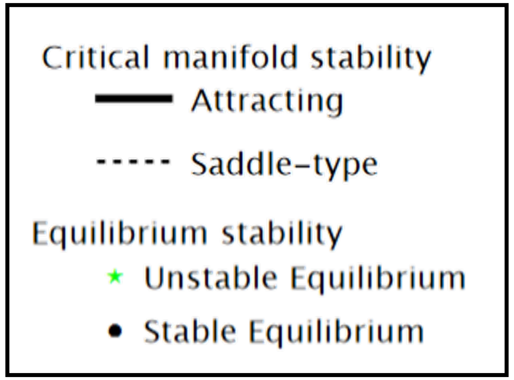
Methodology & Representation

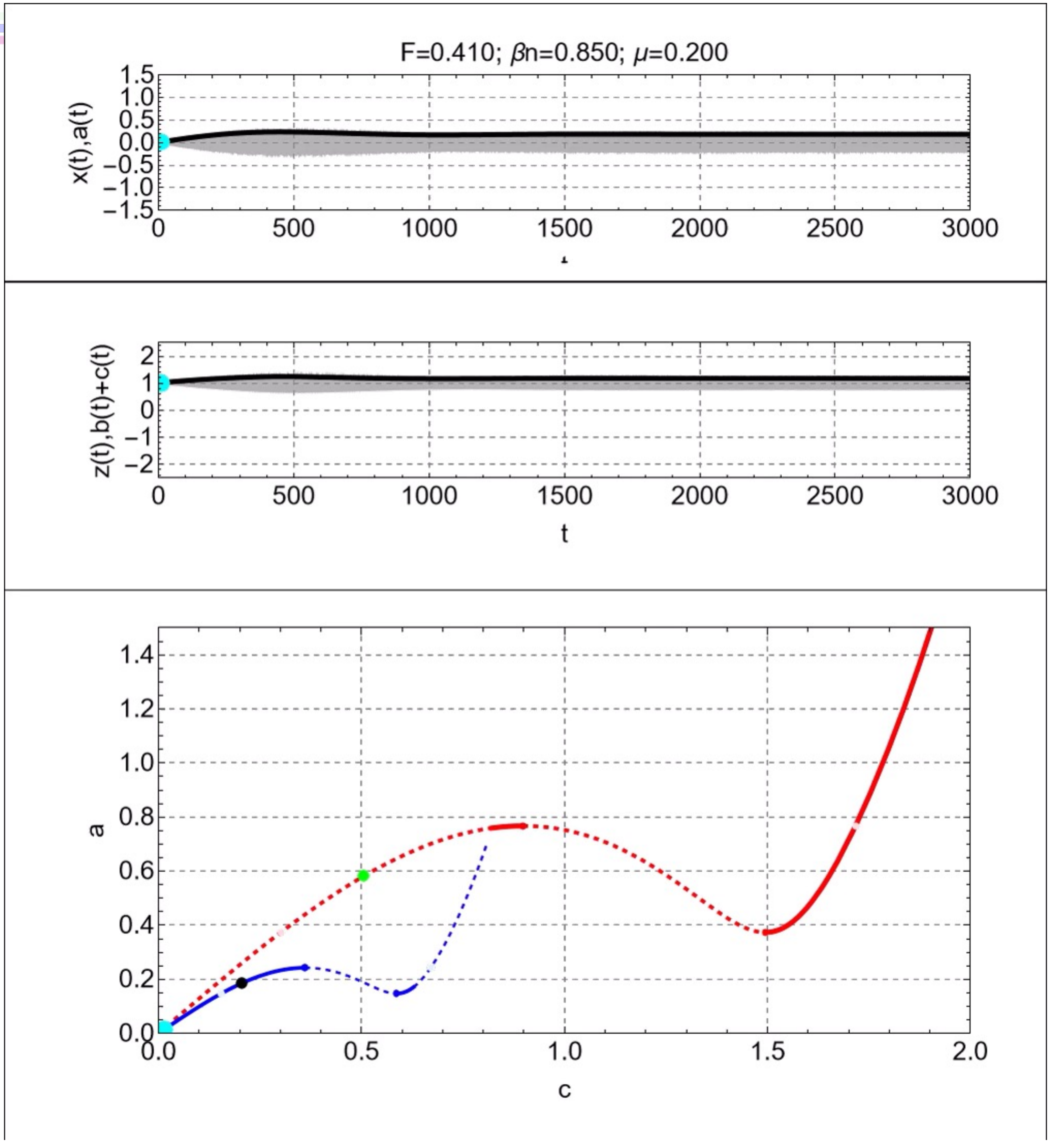
Critical Manifold $\mathcal{M}_0$

Equilibria of the modulation flow:

- To characterize the slow-phase
- Stable equilibrium = possible periodic solution (original dynamic)

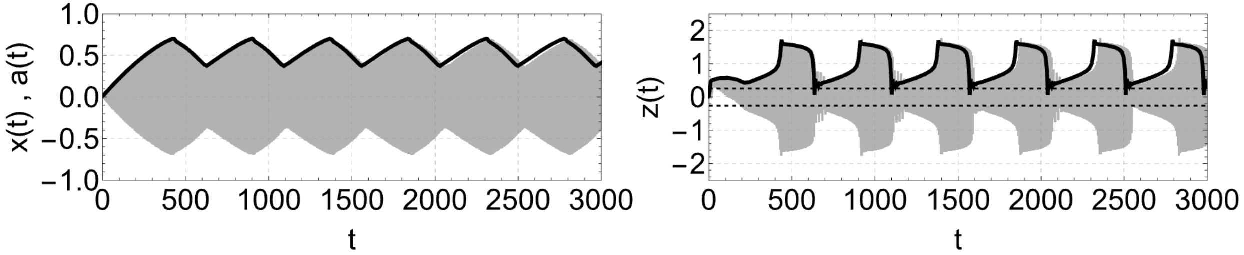




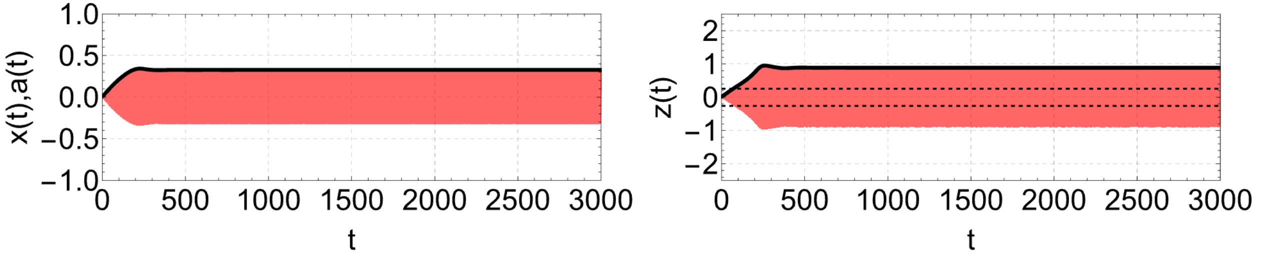


PE influence on Critical Manifold

WITHOUT Piezoelectric element
SMR



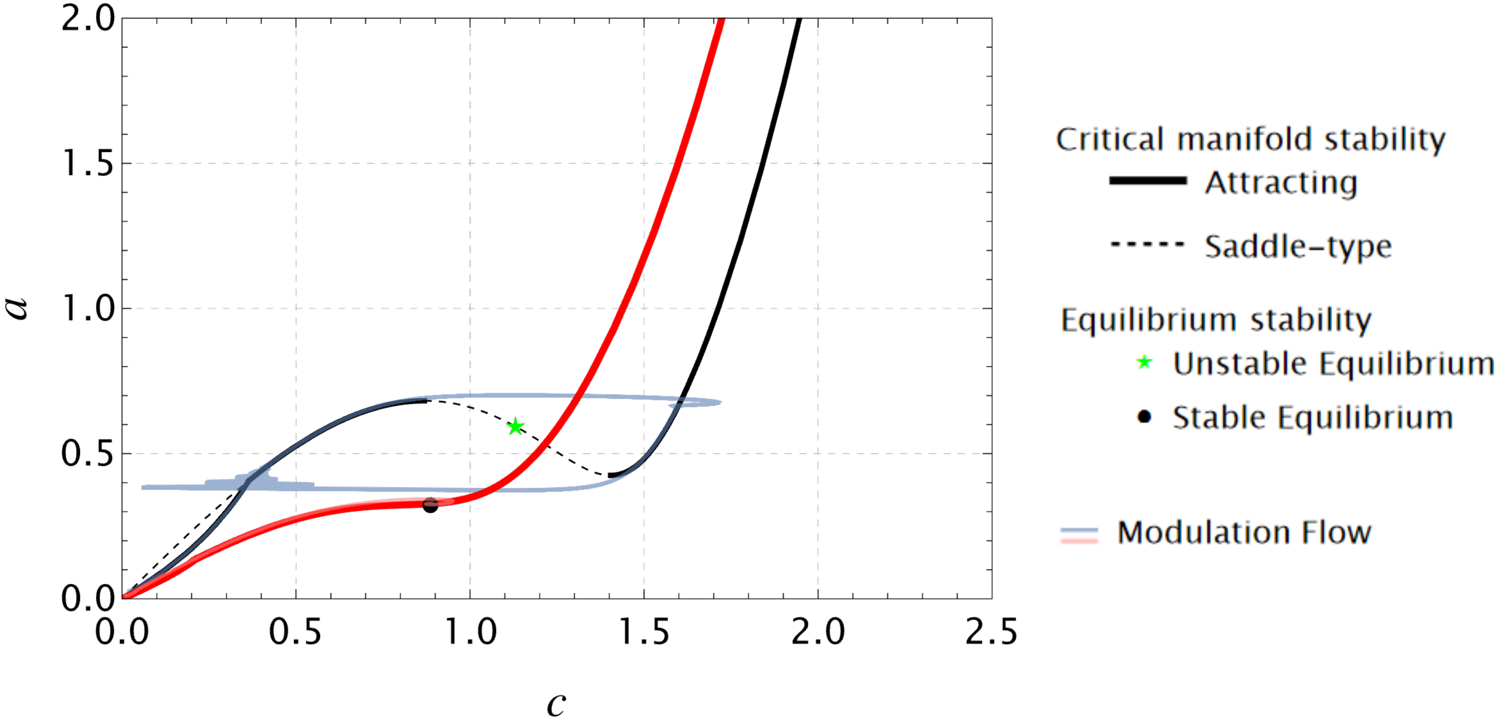
WITH Piezoelectric element
Periodic



$\epsilon = 0{,}005$; $\eta = 0{,}5$; $\sigma = 0$; $\alpha = 0{,}75$;
$F = 1$; $\mu = 0.30$

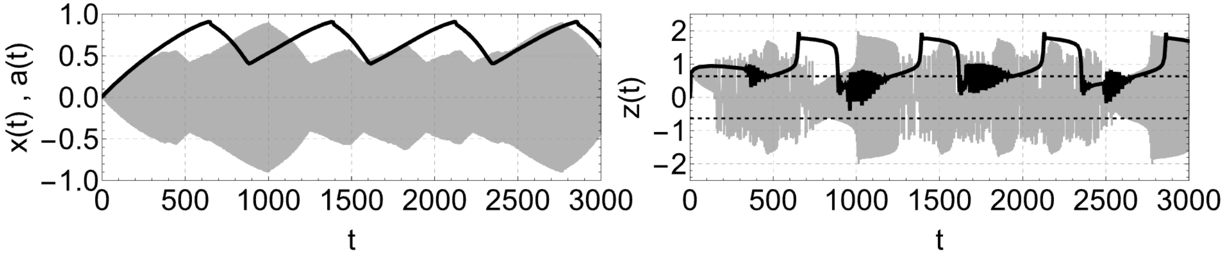
$\beta_p = 0{,}1$; $r = 0{,}1$; $\chi_1 = 0{,}7$; $\chi_2 = 0{,}7$

$\beta = 0.15$

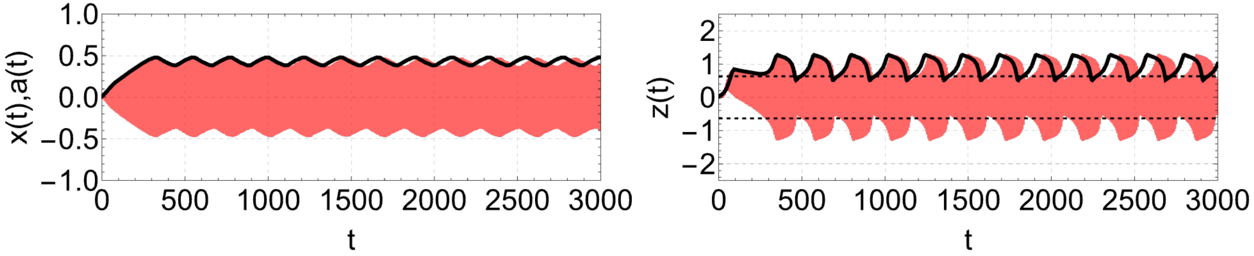


PE influence on Critical Manifold

WITHOUT Piezoelectric element
Chaotic - SMR

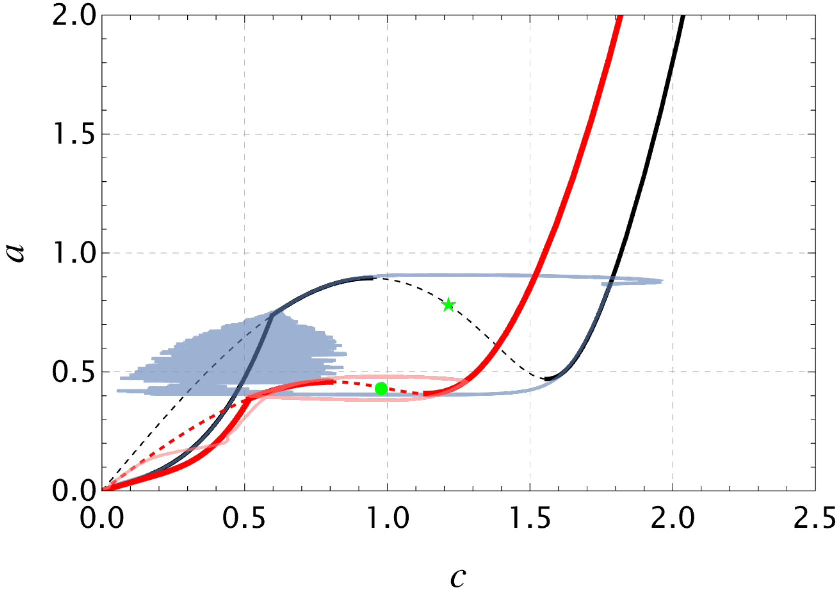


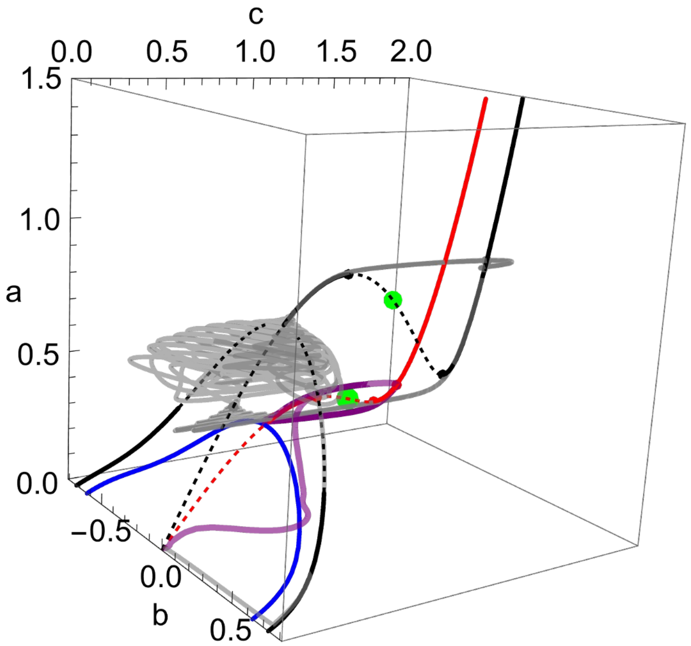
WITH Piezoelectric harvester
SMR



$\epsilon = 0,005$; $\eta = 0,5$;
$\sigma = 0$; $\alpha = 0,75$;
$F = 1$; $\mu = 0.30$;
$\beta = 0.40$

$\beta_p = 0,1$; $r = 0,1$;
$\chi_1 = 0,7$; $\chi_2 = 0,7$



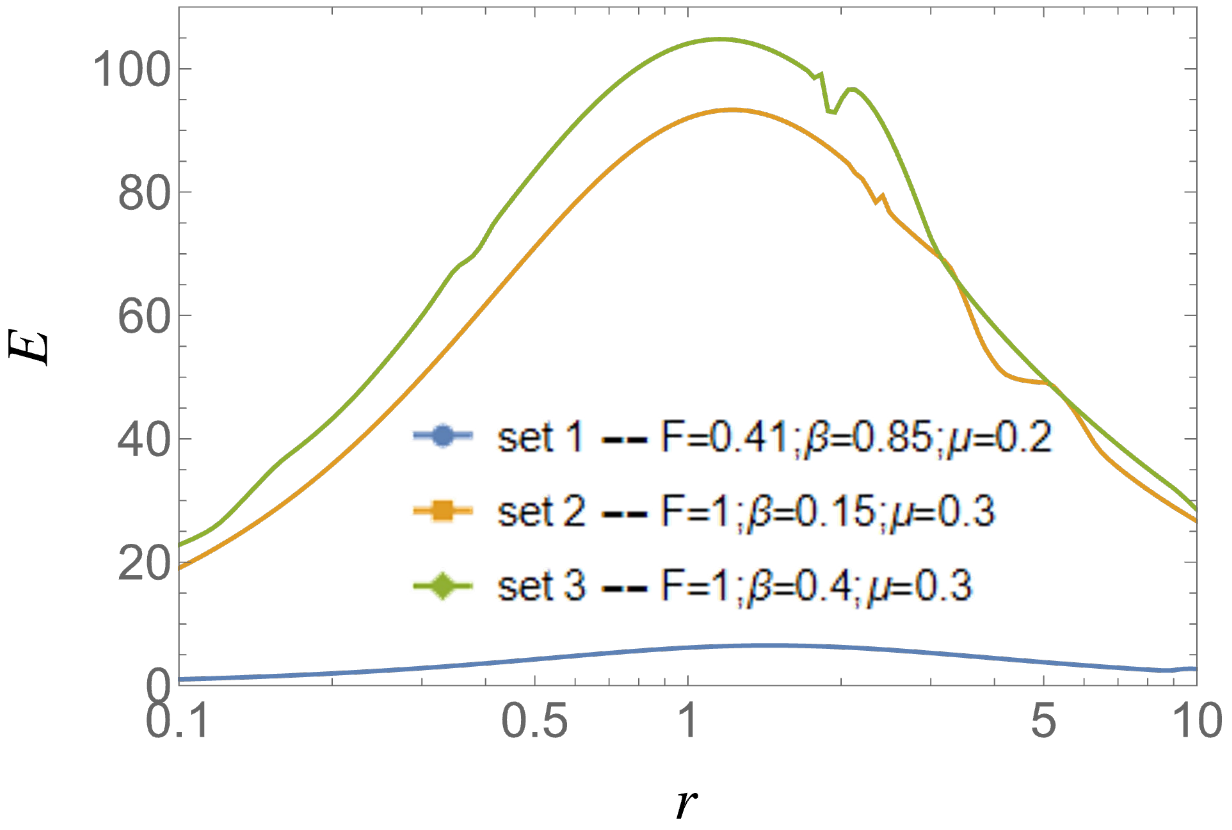


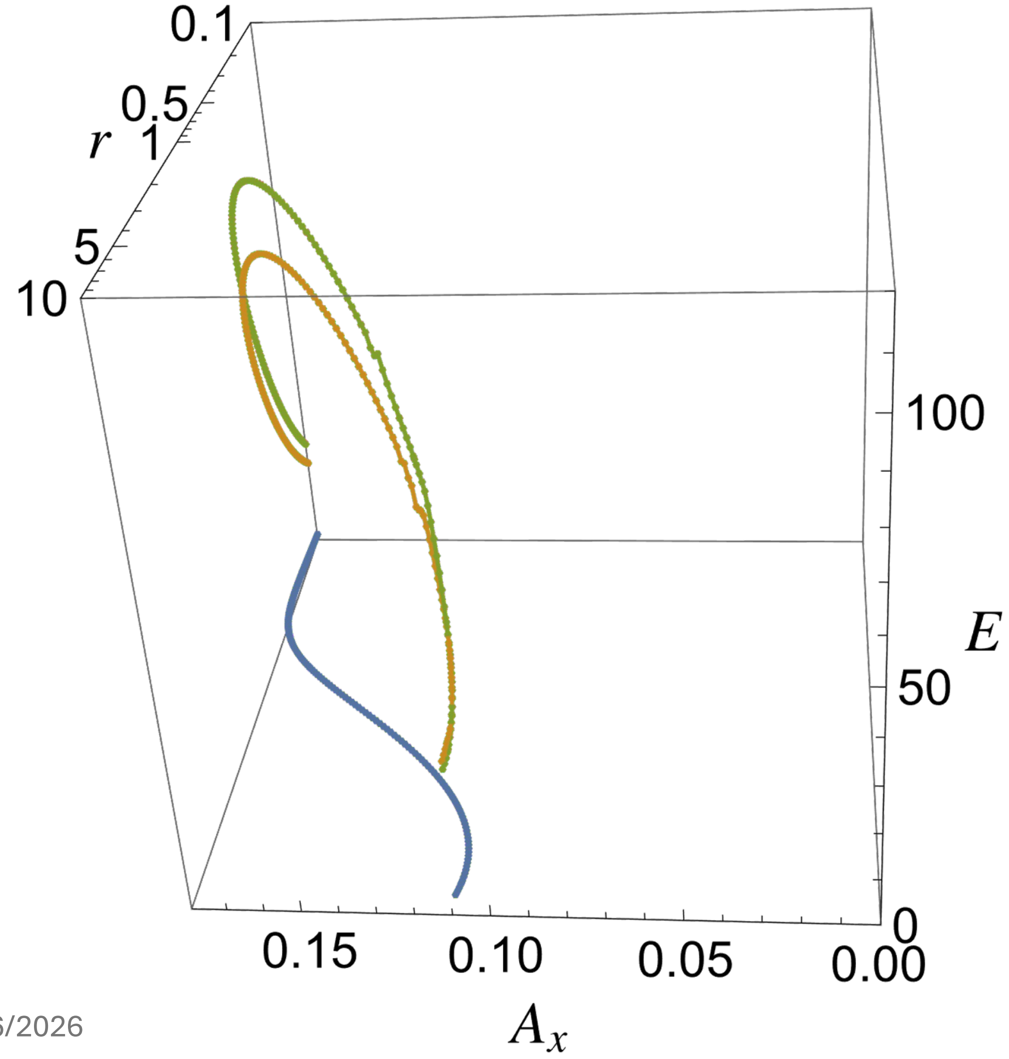
Influences on PEH parameters

Dimensionless harvested energy

$$E = V(t)^2 \, r \, t$$

$$r \propto 1/R$$





Conclusion

- **The method allows to:**
  - **Predict** (or **understand**):
    - the rich dynamics of the system
    - Interpret the effect of the PEH parameters on this dynamics
- **Limits of the method:**
  - **Basins of attraction** need to be investigated for full prediction
  - Pertinence of the 1:1 resonance capture assumptions for chaotic regimes
- **Perspectives:**
  - Comparison of the model with experimental data